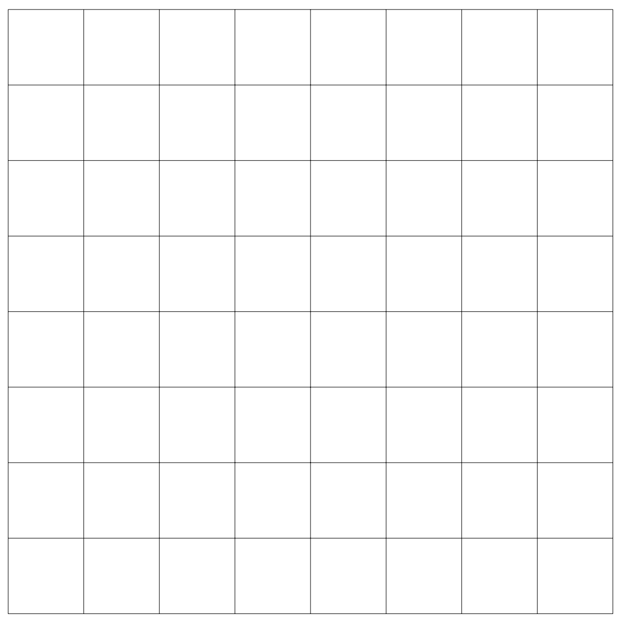
UW Math Circle
Christmas Auction

December 15, 2016

1. What is the largest possible k so that you can place k white queens and k black queens on an 8×8 chessboard so that no piece is attacking any other piece?

   Please neatly draw your final answer either here or on a piece of graph paper.



2. Given a number n, start a sequence $a_1, a_2, \dots, a_m$ where $a_1 = n$, and $a_k = 3a_{k-1} + 1$ if a_{k-1} is odd, and where $a_k = a_{k-1}/2$ if a_{k-1} is even. In other words, you want to divide by two but if your number is odd you can't, so you multiply by 3 and add 1, and now you can divide by two. If your sequence ever reaches one, you stop.

   For example, if I started with 5 my seqeuence would be: $5, 3 \times 5 + 1 = 16, 8, 4, 2, 1$. If I started with 12, my sequence would be $12, 6, 3, 3 \times 3 + 1 = 10, 5, 3 \times 5 + 1 = 16, 8, 4, 2, 1$.

   Many people think that whatever number you start with, this sequence will always end in a one– but this is currently unknown!

   Your task is to find a number less than 1,000 so that the sequence you get when starting with your number contains as large a number as possible. For example, in both of the two examples above, the largest number is 16.

3. If we have numbers $1, 2, \dots, n$, we say that a smaller collection of these numbers generates the entire collection if every number in the bigger set is either in the smaller set or is the sum of two different numbers in our smaller set.

For example $\{1, 2, 4, 7, 10\}$ generates $1, 2, \ldots, 10$, because 1 and 2 are in the set, $1 + 2 = 3$, 4 is in the set, $1 + 4 = 5$, $2 + 4 = 6$, 7 is in the set, $1 + 7 = 8$, $2 + 7 = 9$, and 10 is in the set.

Your goal is to find a generating set for $1, 2, \ldots, 100$ that contains as few numbers as possible.

4. The three numbers $(3, 4, 5)$ are interesting because $3^2 + 4^2 = 5^2$. Another three numbers with this property are $(20, 21, 29)$. Your goal is to find a sequence of positive integers (a, b, c), with $0 < a < b < c < 1000$, with $a^2 + b^2 = c^2$, and where a, b, and c have no common factors, and where $(b - a)(c - b)$ is as a large as possible.

   If my sequence was $(3, 4, 5)$, then my quantity would be $(4 - 3)(5 - 4) = 1$, and for $(20, 21, 29)$ it is $(21 - 20)(29 - 21) = 8$.

   Warning: Giving one of these sequences, it is easy to generate another by multiplying all the numbers by some fixed number. For example, $(3 \times 100, 4 \times 100, 5 \times 100)$ is a sequence satisfying $(3 \times 100)^2 + (4 \times 100)^2 = (5 \times 100)^2$. However, it doesn't satisfy the requirments of the problem since $(3 \times 100, 4 \times 100, 5 \times 100)$ all share the common factor of 100.

5. You have an 11×11 board, and you want to tile it with L shaped tiles and with with lightning shaped tiles, as below. You can rotate and reflect the tiles. What is the tiling that uses the fewest possible tiles?

   Please neatly draw your final answer either here or on a piece of graph paper. Consider using different colors for the different tiles.