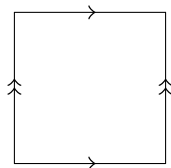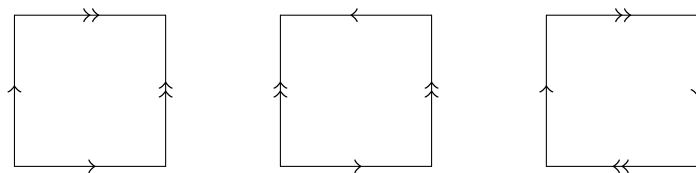# 1   Torus and Klein Bottle

A torus can be constructed from a square by identifying the edges in the following way:

Such a diagram is called a polygonal presentation of a surface. A sphere, Klein Bottle and Projective-plane can be constructed in the three following ways:

1. A Cat's game in Tic Tac Toe is a game where neither side wins, even though the board is filled up with X's and O's. Is it possible to have a Cat's game in Tic Tac Toe on the torus?

2. How many essentially different opening moves does the first player have in Tic Tac Toe on the torus? How many different responses does the second player have?

3. Is there a winning strategy for the first player in Tic Tac Toe on the torus? That is, is it possible for the first player to win no matter what the second player does? Does it change your answer if the first player is required to start on the center square?

4. Play Tic Tac Toes on the Klein Bottle with a friend. Is it possible to get a Cat's game? Is there a winning strategy for the first player?

2   More constructions

**5.** What do you get when you cut a Mobious band around the center circle?

6. What do you get when you cut a Klein Bottle in half? Does it depend on which way you cut it?

7. Which two surfaces are obtained by gluing the edges of each triangle as shown? You get two different surfaces, one for each triangle. (Hint: for the

second triangle you might consider cutting the triangle before identifying the edges and then repairing the cut.)

8. What surfaces do the following constructions yield?
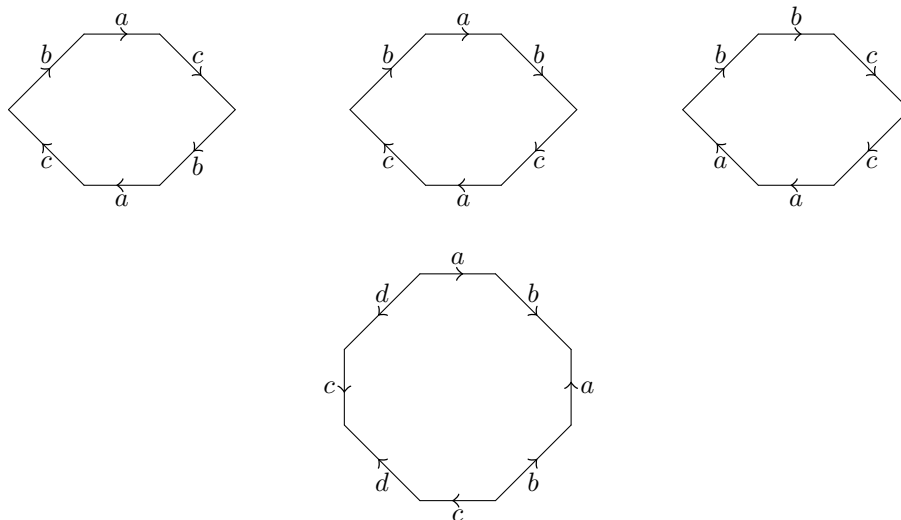
3   Connected sums

The connected sum of two surfaces if constructed by cutting a circle out of each and then gluing them together along this circle. It might be helpful to translate that construction into one involving polygonal presentations to answer the following questions.

9. What do you get when you take the connected sum of two projective planes? (Hint: First, figure out what you get when you remove a disk from a projective plane. Then figure out what you get when you glue two of these together.)

10. Show that the connected sum of the torus and the projective plane is the same as the connected sum of a Klein bottle and a projective plane.