UW Math Circle

10 March 2016

Recall an equivalence relation $\sim$ satisifies the following three properties:

- (Reflexivity) For all a, $a \sim a$.
- (Symmetry) If $a \sim b$, then $b \sim a$.
- (Transitivity) If $a \sim b$ and $b \sim c$, then $a \sim c$.

If $\sim$ is an equivalence relation over X, write $X/\sim$ as the set of equivalence classes of $\sim$ over X. Denote an equivalence class containing a by $[a]$.

1. For $\sim$ defined below, is $\sim$ an equivalence relation? Assume $a, b \in \mathbb{R}$ unless stated otherwise. If so, determine the set of equivalence classes.
   (a) $a \sim b$ if $ab = 0$.
   (b) $a \sim b$ if $ab \neq 0$.
   (c) Suppose a, b are integers. Let $a \sim b$ if $a - b$ is a multiple of 5.
   (d) $a \sim b$ if $|a - b| < 5$.
   (e) Let $f(x) = x^2 + 1$. Let $a \sim b$ if $f(a) = f(b)$. What if $f(x) = x^3 + 1$?

2. Let's expand on 1c) a little more. Often when we define equivalence classes, we take some big set (in this case $\mathbb{Z}$) and define equivalence when two things differ by a multiple of something. So let $a, b \in \mathbb{R}$ and let $a \sim b$ if $a - b$ is a multiple of 1. What are the equivalence classes? What if we replace 1 with $\sqrt{2}$? $\sqrt{\pi}$?

3. Let's expands on 1e) a little more. Say f is a function (for sake of simplicity, say it is a polynomial). What conditions must you impose on f such that 1e) is an equivalence relation?

4. Prove that if $[a], [b] \in X/\sim$ and $[a] \cap [b] \neq \emptyset$, then $[a] = [b]$.

5. Working over the reals again, let $a \sim b$ if $a \leq b$. Is $\sim$ an equivalence relation? No? Why not? In fact, we've defined something called a *partial order* $\triangleleft$. This also satisfies some properties similar to those for an equivalence relation:
   - (Reflexivity) For all a, $a \triangleleft a$.
   - (*Anti*-Symmetry) If $a \triangleleft b$ and $b \triangleleft a$, then $a = b$.
   - (Transitivity) If $a \triangleleft b$ and $b \triangleleft c$, then $a \triangleleft c$.

6. Working over integers, let $a \triangleleft b$ if a divides b. Is $\triangleleft$ a partial order?