UW Math Circle

31 March 2016

1. Find the continued fraction for $\sqrt{2}$.

$$\pi = 3+\cfrac{1}{7+\cfrac{1}{15+\cfrac{1}{1+\cfrac{1}{292+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{14+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{84+\cfrac{1}{2+\cfrac{1}{1+\cdots}}}}}}}}}}}}}}}}}}}}}}}$$

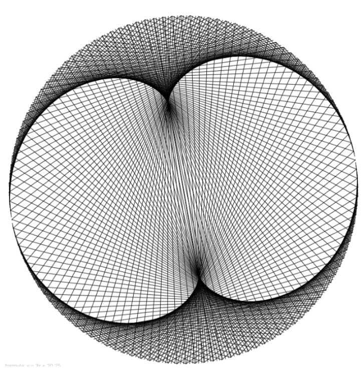
2. Recall that $\pi(n)$ counts the number of primes less than or equal to n. Compute all n such that $\pi(n) > n/3$. (Hint: look mod 6)

3. Prove that n is its own inverse mod p if and only if either $n = 1$ or $n = p - 1$.

4. Prove *Wilson's Theorem*: p is prime if and only if $(p-1)! \equiv -1 \pmod{p}$.
   Some points to help you:
   - In $(p-1)!$, pair up each number with its inverse mod p.
   - Don't forget to prove the other direction: if p is not prime, then $(p-1)! \not\equiv -1 \pmod{p}$. In fact, it is equal to $0 \pmod{p}$.

5. Refresh your own memory on the definition of polynomial division.

6. Practice with modular arithmetic on polynomials:
   - What's the degree of $(2x^2 + 3x + 2)(2x^3 + 3)$ in $\mathbb{Z}_4[x]$?
   - Compute $(x^3 + x + 1)/(2x + 1)$. Then compute it again in $\mathbb{Z}_3[x]$.
   - Let $\deg(f) = A, \deg(g) = B$. What is the best statement you can make about $\deg(f + g)$?
   - How many polynomials of degree at most d are there in $\mathbb{Z}_n[x]$?



7. A set of polynomials is called an *idea* if it is closed under addition and absorbs multiplication. List five ideas in $\mathbb{Z}[x]$.