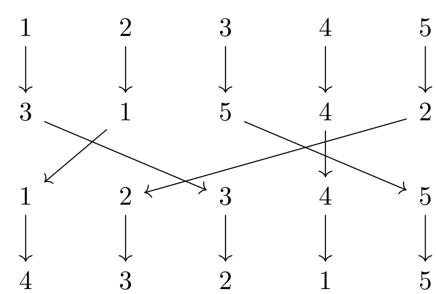
UW Math Circle

December 1, 2015

When we examined the symmetry groups of an n-gon we saw that the elements of our group were actually functions and the group operation was composition. Today we talked about a new group, the group of permutations of n objects. This group is called the symmetric group and is denoted S_n. A permutation n objects is a function from the set $\{1, \ldots, n\}$ to itself which does not send any two numbers to the same place.

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 1 & 5 & 4 & 2 \end{array}$$

For notational purposes we denote the permutation above by 31542, i.e. send 1 to 3, send 2 to 1, and so forth. To compose two functions we first do one and then the other. We can do this graphically.



Here we are computing $21542 \circ 43215$. Going from top to bottom we see that $1 \mapsto 2$, $2 \mapsto 4$, $3 \mapsto 5$, $4 \mapsto 1$ and $5 \mapsto 3$ which is the permutation 24513.

Problem 1: Show that the set of permutations is in fact a group. How many elements are in S_n?

To understand the symmetric group we often use another form of notation called cyclic notation. This notation takes advantage of the fact that when we repeatedly apply a permutation it will send the numbers in a "cycle". For example, if we repeatedly apply the the permutation 32541 we see that 1 gets sent to 3 which is sent to 5 which is sent to 1. Applying this permutation again will just continue this cycle. Similarly, 2 is sent to 4 which is sent back to 2. In cycle notation we write 32541 as (135)(24). Another example is 4213, an element of S_4 which is written as (143)(2). We call 2 a fixed point.

Problem 2: Show that every permutation can be written using cyclic notation.

Problem 3: Recall that the order of an element is the fewest number of times you must compose it with itself before you get back to the identity. What is the largest order of any element in S_4? What about S_5? How many elements in S_4 have order 3? How many elements in S_9 have order 15? How many elements in S_9 have order 6? (Hint: for the last trhee parts it will help to remember how to count the number of ways to choose k things from n objecst.)

We have previously seen two other families of groups. The cyclic group, C_n was written as the set $\{0, \ldots, n-1\}$ with addition modulo n as the group operation. The dihedral group, D_n is the symmetries of the n-gon and we generated it with flips and rotations. From their constructions we can count that C_n contains n elements, D_n contains $2n$ elements and S_n contains $n!$ elements.

Problem 4: Notice that C_6, D_3 and S_3 all have 6 elements. Are any of these groups the "same"? Can you show that these groups are "different"? We can ask the same questions for C_{24}, D_{12} and S_4. (Hint: we have not defined what it means for two groups to be the "same" or "different" but you should still be able to find some similarities and differences. If two groups are the same they will have the same multiplication tables. Difference are easier to spot, you can check properties like being commutative.)