UW Math Circle

December 10 2015

Here are a bunch of problems. The problems in Section A should be thought of as warm ups; you do not need to present these. Section B is the main focus of the problem set. These are problems similar to those found on other worksheets. You should work mostly on these and hopefully solve a handful. Section C problems are a significant increase in difficulty. You should only do these if you are comfortable with the previous problems and are ready to tackle advanced topics.

A

1. Determine whether the following are functions:
   (a) $f : \mathbb{R} \to \mathbb{R}, f(x) = \sqrt{x}$
   (b) $f : \mathbb{Q} \to \mathbb{R}, f(x) = x^2$
   (c) $f : \mathbb{R} \to \mathbb{Q}, f(x) = x^2$
   (d) $f : \mathbb{Q} \to \mathbb{Q}, f(x) = x + 1$
   (e) $f : \mathbb{Z} \to \mathbb{Q}, f(x) = x/2$
2. For the functions above, determine whether they are injective or surjective.
3. For the functions above, determine their range.
4. Explain why division usually cannot be a group operation.
5. Explain why $\mathbb{Z}$ is not a group under multiplication, even if we remove 0.
6. Let $a \star b = (a + b)/2$. Is $(\mathbb{Q}, \star)$ a group?
7. Let $a \star b = 0$. Is $(\mathbb{Q}, \star)$ a group?
8. Find three subgroups of $\mathbb{Z}$. Find three subgroups of $\mathbb{R}$ that contain $\mathbb{Z}$.
9. Find the smallest group that contains $\mathbb{Z}$ as well as $\sqrt{2}$.
10. If $k < n$, find a subgroup of S_n that has exactly k elements.

B

1. (a) Find a function $f : \mathbb{Z} \to \mathbb{Z}$ that is injective but not surjective.
   (b) Find a function $f : \mathbb{Z} \to \mathbb{Z}$ that is surjective but not injective.
2. Prove that the intersection of two subgroups is also a subgroup.

3. Find all subgroups of C_9, C_{10}, and C_{11}.
4. If $f : G \to H$ is a group homomorphism, and G is cyclic, prove that H is cyclic as well.
5. Find all homomorphisms from $C_{10} \to C_4$. How about from $C_4 \to C_{10}$?
6. Prove that H is a subgroup of G if and only if a and b in H implies ab^{-1} is in H as well.
7. Prove that the dihedral group is a subgroup of the symmetric group.
8. Find all groups with 4 elements.
9. How many elements in S_{10} have order 21?
10. Find the highest possible order of an element in S_{14}.

C

1. (a) Prove that if $f : A \to B$ and $g : B \to C$ and $g \circ f : A \to C$ is surjective, then g is surjective. Is f surjective?
   (b) Prove that if $g \circ f : A \to C$ is injective, then f is injective. Is g injective?
2. Let H be a subgroup of an abelian group G.
   (a) For $g \in G$, let gH be the set of elements gh, where h is an element of G. If $g \in H$, show that $gH = H$.
   (b) If g was not in H, show that gH and H don't overlap at all.
   (c) Show that gH has the same number of elements as H.
   (d) For g_1 and g_2 in G, show that g_1H and g_2H are either the same or don't overlap at all. Such sets gH are called *cosets* of H in G.
   (e) Recall that G was abelian. Prove that the set of cosets is a group. This is called the *quotient group* G/H and is an extremely important algebraic topic.
3. In the above problem, suppose G was not abelian. Then we can define a different type of coset: Hg instead of gH. Determine the condition necessary on H for these two definitions to cosets to be same. If H satisfies this condition, it is called a *normal* subgroup. Prove that all subgroups of an abelian group are normal.
4. Let $f : G \to H$ be a group homomorphism. Let S be the subset of G that f maps to e_H. Show that S is a subgroup of G. Furthermore, show that S is a normal subgroup. S is called the *kernel* of f in G.
5. Show that the subgroup of a cyclic group is cyclic.