UW Math Circle (Invariants)

April 21st, 2016

1. Begin with a stack of chips. On your turn, you may take either 2 or 3 chips from the stack. If you take all the chips, you win; if there's 1 chip left, you lose. For what starting numbers of chips is there a winning strategy?

   What if you are only allowed to take 3 or 4 chips? What if you are only allowed to take 5 or 11 chips? (In any case, if there's a pile of chips left at the end, you lose, and if you take them all, you win).

2. In the game of squares and circles, begin with a collection of squares and circles. For example, you might start with three circles and a square. At each step, cross out any two shapes. If the shapes you just crossed out are the same, draw one square. If they are different, draw one circle. Eventually, there is only one shape left. You win if the final shape is a circle. What can you say about what happens in this game? What about with different starting situations?

3. In the previous problem, what if we have squares, circles, and triangles, and the rule is that you may only cross out two *different* shapes and then draw the third shape. Can you win (by ending with just one circle) if you start with three circles and a square?

   What about other starting situations, like four of each shape for example?

   What about if you can reverse the rule when you wish, crossing out *one* shape and drawing one each of the other two shapes?

   What if the rules are changed so that when you cross out two different shapes you then draw *two* copies of the third shape? The goal is to get all the shapes the same. What starting situations enable you to eventually win?

4. The Mad Veterinarian has three machines. One converts a cat into two dogs and a mouse (or vice-versa): $1C \leftrightarrow 2D, 1M$. A second machine does $1D \leftrightarrow 1C, 1M$, and a third machine does $1M \leftrightarrow 1C, 3D$. The general puzzle is to start with just one animal and replicate it: what's the fewest cats (more than one) that you can turn one can into (with no mice or dogs left around)? Or, even more generally, starting with one cat can you describe all the combinations of animals you can end up with?

5. Here's a two-player game for a change: Player 1 writes a sequence of ten positive integers. Then player 2 writes a + or − sign in each of the nine spaces between the integers. In the end, if the final numeric result is odd, player 1 wins, and if even, player 2 wins. Who should win this game, and how?

   What if player 1 is given a bag with a certain collection of numbers, each of which can only be used once? For example, if they have a bag containing the numbers 1 through 12? 1 through 11? 1 through 10?

   What if player 2 can use exactly one multiplication sign, and eight + or − signs, in the nine spaces?

   What if player 2 gets exactly two multiplication signs?

6. Another one-player game: Start with a stack of n boxes. At each move, as long as any stacks have more than one box, split one stack into two parts, say x boxes into y and z, and score yz points. How should you split them in order to maxemize your score? What is the maximum score for each n?

7. Coin-flipping: Begin with some number of coins, say four for example, and set them on the table in a line, with a given starting sequence like $HHTH$ for example. At each move, you may flip any two adjacent coins. You win if the final arrangement of the coins is all heads.

8. Coin-splitting: Begin with an infinite strip of squares, and a penny on one spot. At each move, you may either split the penny (remove it an put a penny on each adjacent spot) or merge two pennies (remove two pennies with exactly one space between them and put one on the space between; in other words, undoing the splitting operation). You may have any number of pennies on a given spot (but each move only splits one penny or merges two pennies into one). Starting with one penny, can you split and merge to end up with just one penny on the board in a different spot? What different spots are possible?