UW Math Circle (Graphs)

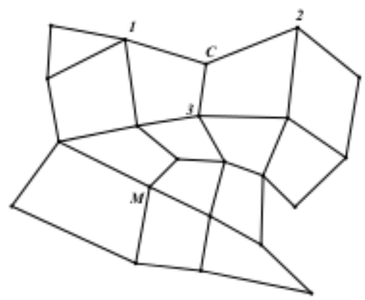
May 26th, 2016

1 Three Games

For games 1 and 3, two players alternate turns. The rule for a legal move is described. The game ends when no legal moves can be made. The winner is the last player to make a legal move. Your job is to analyze the game and figure out a winning strategy.

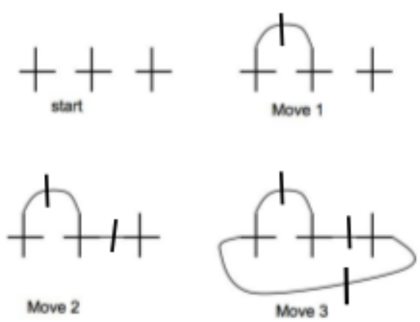
1. *Color the Grids.* You start with an $n \times m$ grid of graph paper. Players take turns coloring red one previously uncolored unit edge of the grid (including the boundary). A move is legal as long as no closed path has been created.

2. *Cat and Mouse.* A very polite cat chases an equally polite mouse. They take turns moving on the grid depicted below.



   Initially, the cat is at the point labeled C; the mouse is at M. The cat goes first, and can move to any neighboring point connected to it by a single edge. Thus the cat can go to points 1, 2, or 3, but no others, on its first turn. The cat wins if it can reach the mouse in 15 or fewer moves. Can the cat win?

3. *Brussels Sprouts.* Start by putting a few crosses on a piece of paper. On each move, a player can connect the two endpoints of a cross together, with a single line (which can be curved). Then a new cross is drawn on this connection line. You cannot ever draw a line that intersects another already-drawn line. Here is an example of the first few moves of a 3-cross game.



2 Some useful tools

Try to prove as many of the following properties that you can:

1. *Handshake Lemmma.* The sum of the degrees of the vertices equals twice the number of edges; as a corollary, if v is odd, one of the vertices has even degree.

2. For connected graphs, $e \geq v - 1$, with equality holding for trees. For a forest with k connected components, $e = v - k$.

3. If $e \geq v$, then the graph has a cycle.

4. A graph is bipartite if and only if it has no odd cycles.

3 Puzzles

1. Show that every graph contains two vertices of equal degree.
2. In the nation of Klopstockia, each province shares a border with exactly three other provinces. Can Klopstockia have 17 provinces?
3. Draw a graph with eight vertices, four of which have degree 4 and four of which have degree 3.
4. Show that it is possible to have a 4-regular graph with n vertices, for every $n \geq 5$.
5. If 127 people play in a singles tennis tournament, prove that at the end of the tournament, the number of people who have played an odd number of games is even.
6. How many edges must a graph with n vertices have in order to guarantee that it is connected?
7. A large house contains a television set in each room that has an odd number of doors. There is only one entrance to this house. Show that it is always possible to enter this house and get to a room with a television set.
8. Show that if a graph has v vertices, each of degree at least $v/2$, then this graph is connected. In fact, show that it is Hamiltonian.
9. A tournament is a directed graph in which every pair of vertices occurs as an edge in one order or the other (but not both). Prove that every tournament has a (directed) Hamiltonian path. Also, which tournaments contain a Hamiltonian cycle?
10. During a certain lecture, each of five mathematicians fell asleep exactly twice. For each pair of these mathematicians, there was some moment when both were sleeping simultaneously. Prove that, at some moment, some three of them were sleeping simultaneously.

4 Definitions

1. If the vertices of a graph can be partitioned into two (non-empty) subsets such that all edges of the graph connect only vertices from different sets (never two vertices from the same sets), then the graph is called **bipartite**.
2. A walk with no repeated edges is called a **trail**. A walk with no repeated vertices is called a **path**.
3. A closed trail is called a **circuit**. A closed path is a contradiction in terms, but what this term evokes is called a **cycle**. More precisely, a cycle is a closed walk in which no vertex is repeated except for the starting vertex (which is the same as the end vertex).
4. An **Eulerian trail/circuit** is a trail/circuit which visits every edge of a graph. Such a graph is called **Eulerian**.
5. A **Hamiltonian path/cycle** is a path/cycle which visits each vertex of the graph. Such a graph is called **Hamiltonian**.