UW Math Circle

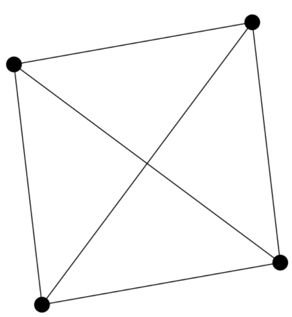
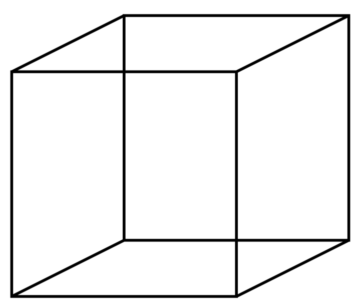
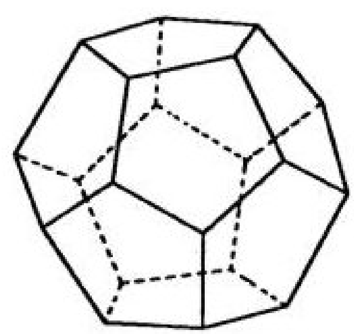
February 11, 2016

1. We call a graph planar if we can draw it in the plane without any of the edges crossing. A face of a planar graph is a region bound by the edges. Note that the region outside a graph is also a face.

   (a) What is the minimum number of edges bounding a face for a graph with more than three edges and without multiple edges between two vertices?

   (b) For a planar graphs without multiple edges between vertices and with more than 3 edges, showsthat 2# of edges $\geq$ 3# of faces.

2. Which of the following graphs are planar?

3. We call a graph Eulerian if there is a closed path (meaning it starts and ends at the same vertex) around the graph that visits each edge exactly once.

   (a) Show that every vertex in an Eulerian graph has even degree.

   (b) Show that if a graph is connected (meaning you can get from one vertex to every other vertex by traveling along paths) and every vertex has even degree, then it is Eulerian.

4. In this problem we will prove Euler's formula for graphs, which says that for a planar graph, $\#\text{vertices} - \#\text{edges} + \#\text{faces} = 2$. We'll call these quantities V, E, and F.

   (a) For an Eulerian graph G, and a closed path in the graph that visits each edge exactly once, let R be the number of vertices that are repeated. For example, if the graph was just vertices and edges arranged in a circle R would be one (the repeated vertex is the starting and ending vertex). Show that $F = R + 1$, and that $R = E - V + 1$. Conclude that for Eulerian graphs, $V - E + F = 2$.

   (b) Using the first part of this problem, show that $V - E + F = 2$ is true for any planar graph.

5. For a planar graph without multiple edges between two vertices, show that the average degree of a vertex is less than 6.

6. Show that we can color any planar graph with six colors, where two vertices have different colors if they share an edge.