Practice Olympiad

UW Math Circle

May 28 2015

1. In a group of people, people are either friends or strangers with each other. That feeling is mutual, i.e. if I'm your friend, you're my friend. Prove that there are two people that have the same number of friends.

Figure 1: Some friends

2. Prove that any consecutive Fibonacci numbers are coprime.

3. Some people are waiting in line. Prove that they can arrange themselves in any order, just by swapping people who are next to each other.

Figure 2: A long line of people

4. Prove that there are infinitely many primes equal to 1 mod 4.

5. Alice and Bob are playing a game, using a box with n sticks in it. Alice and Bob take turns removing sticks from the box, but no more than one third of the sticks at a time. Whoever cannot move loses. Alice goes first; for what n does Alice win?

Figure 3: A cardboard box, possibly with sticks inside

6. At the movie theater, there will be one movie playing each hour for the next 24 hours. The two choices are either movie A or movie B. Alice and Bob go the movie theater together, but Alice wants to watch movie A and Bob wants to watch movie B. Prove that they can find a 6 hour block that contains 3 showings of Movie A and 3 showings of Movie B.

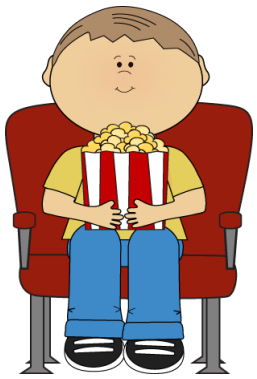

Figure 4: A boy at a movie theater

7. Prove that when $n \geq 6$,
$$\left(\frac{n}{3}\right)^n < n! < \left(\frac{n}{2}\right)^n$$

8. (hard if you weren't here for the lecture on sequences and series) We mentioned in a previous lecture that the harmonic series $1+1/2+1/3+\ldots$ diverged, that is, summed to infinity. But it turns out, if you remove all the terms with the digit 9 in them, the final sum converges, that is, summed to a finite number. We never proved this, but took it as a cool property of series of numbers. Well, now it's your time to prove it. Hint: follow a proof similar to the one which shows that the harmonic series diverges.

9. (quite difficult) Prove that a number n is prime if and only if $(n-1)!+1$ is divisible by n. This is called Wilson's theorem. Hint: Look mod n and use Fermat's little theorem.