Start-of-the-Year Olympiad

UW Math Circle

Session $\omega + 1$ (25 September 2014)

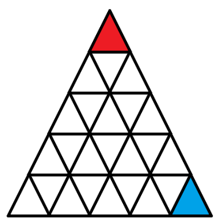
1. (Moscow City MO 2014) An expression is written using only the number 1, the operation signs + and $\times$, and parentheses. The value of this expression is 2014. Is it possible that the value of this expression is still 2014 when all the + signs are replaced with $\times$ signs and vice versa?

2. A 30×50 rectangle is drawn on a sheet of graph paper. Is it possible to draw a line that passes through exactly 80 cells of the rectangle?

3. Take an equilateral triangle with sides of length 2014 and divide it into 2014^2 little (side length 1) equilateral triangles in the "obvious" way (as shown with 5 substituted for 2014). A bug starts in the top corner and wants to crawl to the bread crumb in the right corner following three rules:

   - The bug can crawl from a triangle to any triangle with which it shares an edge.
   - The bug cannot crawl directly up – only down and left/right along a row.
   - The bug cannot visit any triangle more than once.

   How many different paths can the bug take to reach the bread crumb?



4. A robot (who lives strictly in the first quadrant) begins at point $(5, 6)$. It can make two types of moves:

   - From point (a, b), move up, down, left, or right by a or by b.
   - From point (a, b), move to $\left(\frac{a}{2}, \frac{b}{2}\right)$, provided that a and b are both even.

   Could the robot reach the point $(6, 12)$ after a sequence of these moves?

5. (Russia 2009) In a group of seven people, every pair are either friends or strangers. It is possible to seat any six of them at a round table so that everyone is friends with both of their neighbours. Prove that it is possible to seat all seven of them at a round table so that everyone is friends with both of their neighbours.

6. (Leningrad City MO 1983) 2014 unemployed bankers are standing in a circle, and each one starts with an even number of dollar bills. On the count of three, each banker (simultaneously) gives exactly half of their dollar bills to the banker to their right. After this operation, any banker left with an odd number of dollars gets "bailed out" with one additional dollar from the government. This process is repeated once a minute.

   Prove that after some time all the bankers will have the same amount of money.

7. Prove that it is not possible to select $n+1$ diagonals or sides in a convex n-gon so that any two of the selected diagonals or sides intersect (possibly at a vertex). *Hint: By induction on n.*

8. (Moscow City MO) Some mathematicians came to a conference. Every pair were either friends or strangers. Any two of them had exactly two mutual friends among the attendees. Prove that all of the attendees had the same number of friends.

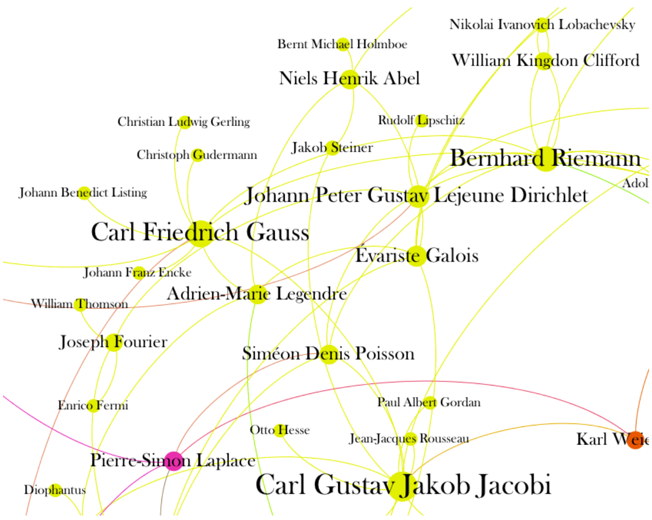


Image: griffsgraphs.com