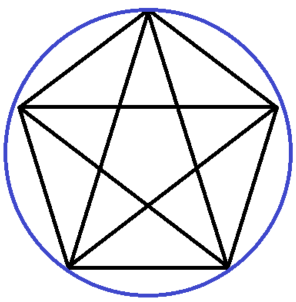
Problem Set 1

UW Math Circle

Session $\omega + 3$ (9 October 2014)

1. Suppose that a is a constructible number. Show that $\sqrt{a}$ is also constructible.
2. The goal of this problem is to construct a regular pentagon with compass and straightedge.
   (a) Find the side length of a regular pentagon inscribed in a circle of radius 1.
   You can do this in two ways:
   - By finding similar triangles in this picture:



   - By expanding $\sin(5\alpha)$ in $\sin\alpha$ and $\cos\alpha$. You can find $\sin(5\alpha) = \sin\alpha \cdot P(\cos\alpha)$, where P is a polynomial of degree 4. Solve $P(\cos\alpha) = 0$ to get two positive solutions for $\cos\alpha$; one of them is $\cos(36°)$ and the other is $\cos(72°)$. Use this to express the side length. (The polynomial P is called the fourth-degree *Chebyshev polynomial of the second kind.*)

   (b) Show that the length you found in (a) is an element of $\mathbb{Q}\left[\sqrt{5}\right]$, so, by the previous problem, it is constructible. Show how to construct this length, and thus how to construct a regular pentagon inscribed in a circle of radius 1.
3. Find the maximum number of regions into which n planes can divide three-dimensional space, knowing that it is a polynomial of degree 3 in n. *(Hint: The polynomial looks simpler when expressed as* $a\binom{n}{3} + b\binom{n}{2} + c\binom{n}{1} + d\binom{n}{0}$ *than when expressed as* $an^3 + bn^2 + cn + d$*.)*
4. (a) (Olimpíada Iberoamericana de Matemática 1991) Each vertex of a cube is labeled with either 1 or -1. Then each face of the cube is labeled with the product of the four numbers at its corners. Is is possible that the sum of all 14 numbers on the cube is 0?
   (b) Same question, except the complex numbers i and $-i$, where $i \cdot i = -1$, are also permitted as labels.