Problem Set 12

UW Math Circle

Session $\omega + 23$ (23 April 2015)

1. Consider the following closure operator defined on the squares of a chessboard.

   Start with a set of squares. If three cells of a little 2×2 square are in your set, add the fourth cell to the set. Repeat until you cannot do this anymore. The set you end with is the closure of the set you started with.

   $$\begin{array}{|c|c|}\hline \times & \times \\ \hline \times & \\ \hline \end{array} \to \begin{array}{|c|c|}\hline \times & \times \\ \hline \times & \times \\ \hline \end{array}$$

   (a) Show that the closure of any set is a union of rectangles that do not touch.

   (b) Is this a closure operator? If it is, what are the corresponding rank function and independence criterion?

   (c) Same as (a) and (b), but for a different operator. This time we add a cell to the set if at least two of its four neighbouring cells are in the set.

   (d) (Problem Set 1, 3 October 2013) A square field is divided into a 10×10 grid of small square plots. Weeds have infested 9 of the plots. The weeds will spread to a plot if two of the plots adjacent to it (i.e., sharing a side) are already infested. It is possible that the weeds will eventually take over the whole field? (Hint: What property of the region infested by weeds is invariant or monovariant when the weeds spread to a new plot?)

2. Fill in the question mark and prove: For every n:

   $$\left[\binom{n}{1} + \binom{n}{4} + \binom{n}{7} + \binom{n}{10} + \dots\right] - \left[\binom{n}{2} + \binom{n}{5} + \binom{n}{8} + \binom{n}{11} + \dots\right] = ?$$

3. What is the minimal size of a square you can tile with squares of side length 1, 2, and 3 if you have to use exactly the same number of squares of each size?