

Things to Think About 4

UW Math Circle, Winter 2013

Below are three problems from previous BAMO exams. Try solving all of them, then choose one to write up (even if you only have a partial solution!). Try to write your solution in the way we discussed in class: concisely state what you are claiming and follow with your proof. As we have seen in class, it is perfectly fine to prove multiple claims before arriving at a proof to the original problem.

Note: these problems are *hard*, so give yourself some time to really think about them. Work on them a little bit now and then think about them when you are on the bus, or have some time to spare at school, or are waiting to be picked up, etc. Remember: the BAMO exam allows 4 hours for 4 problems - don't expect to solve these quickly!

Problem 1: Place eight rooks on a standard 8×8 chessboard so that no two are in the same row or column. With the standard rules of chess, this means that no two rooks are attacking each other. Now paint 27 of the remaining squares (not currently occupied by rooks) red.

Prove that no matter how the rooks are arranged and which set of 27 squares are painted, it is always possible to move some or all of the rooks so that:

- All the rooks are still on unpainted squares.
- The rooks are still not attacking each other (no two are in the same row or same column).
- At least one formerly empty square now has a rook on it; that is, the rooks are not on the same 8 squares as before.

Problem 2: All vertices of a polygon P lie at points with integer coordinates in the plane, and all sides of P have integer lengths. Prove that the perimeter of P must be an even number.

Problem 3: There are many sets of two different positive integers a and b , both less than 50, such that a^2 and b^2 end in the same last two digits. For example, $35^2 = 1225$ and $45^2 = 2025$ both end in 25. What are all possible values for the average of a and b ?

For the purposes of this problem, single-digit squares are considered to have a leading zero, so for example we consider 22 to end with the digits 04, not 4.

Note: If you have any questions about what any of the problems are asking, whether your solutions are correct, or if you are stuck on a problem please do not hesitate to email Alex (alexvas@uw.edu) or Jonah (jonah.ostroff@gmail.com)!