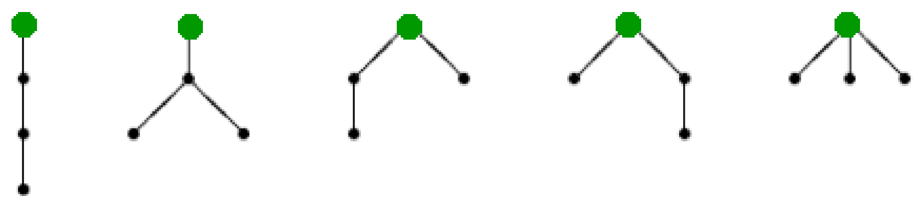
UW Math Circle, Autumn 2013 - Homework 8

Due December 5, 2013

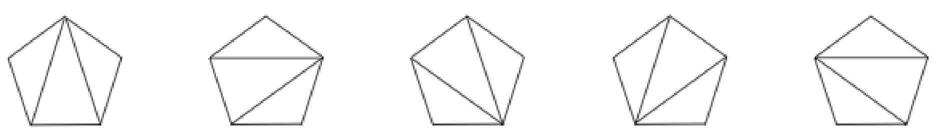
This week, we talked about a lot of types of objects. These (and possibly some other ones) are copied below. For your homework, take any number of pairs of these questions, and prove that they have the same answer. Or, try to find a way of *finding* that answer!

- A **rooted plane tree** is a network of points and edges, where one point is designated the *root*, and every other point is the *child* of some point. Furthermore, all of the children of a given point are ordered from left to right. For example, there are five rooted plane trees with three non-roots:

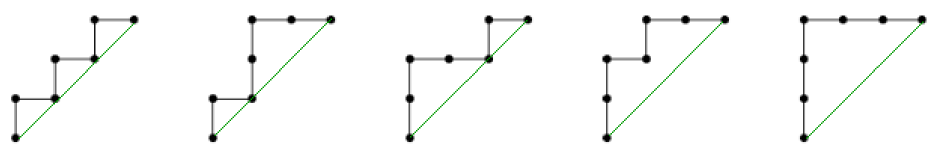
  Question: How many rooted plane trees are there with n non-roots?

- A **polygon triangularization** is a dissection of a polygon into triangles formed by cutting along the vertices, so that no two cuts cross each other. For example, there are five triangularizations of a pentagon:

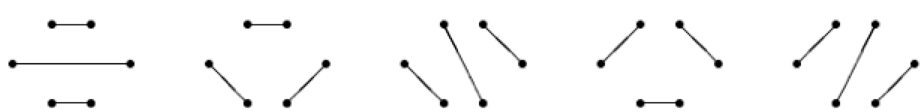
  Question: How many triangularizations are there of an $n+1$-gon?

- A **Catalan path** is a sequence of UP steps and RIGHT steps, starting at $(0,0)$ and ending at (n,n), which never crosses *below* the line $y = x$. For example, there are five Catalan paths from $(0,0)$ to $(3,3)$:

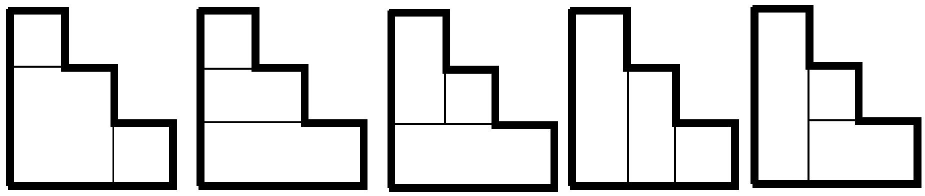
  Question: How many Catalan paths are there from $(0,0)$ to (n,n)?

- $2n$ people sit around a table, and decide to join hands with one another (they can reach very far) so that no two hands cross. For example, with six people, there are five ways they can join hands:

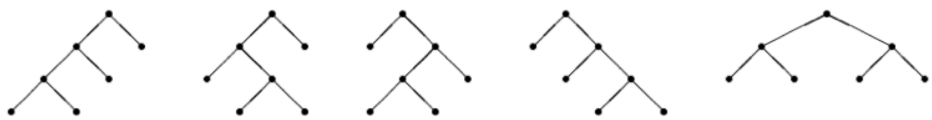
  Question: How many ways can $2n$ people join hands in this manner?

- A **staircase packing** is a way of stacking n rectangles (of arbitrary dimensions) to make a staircase n steps tall. For example, there are five staircase packings using three rectangles:



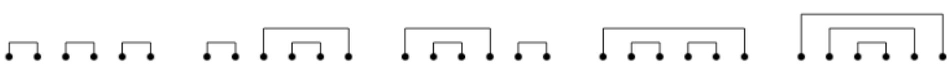
Question: How many staircase packings are there using n rectangles?

- A **binary tree** is like a rooted tree, except every node has either zero or two children. For example, there are five binary trees in which three of the nodes have two children:



Question: How many binary trees are there, in which exactly n nodes have two children?

- $2n$ points are drawn in a line. Then you match those points into pairs and connect them with paths drawn above the line, so that no two paths cross. For example, there are five ways to pair up six points in this way:



Question: How many ways are there to pair up $2n$ points in this manner?

- A permutation of the numbers from 1 to n is **123-avoiding** if no three of the numbers (consecutive or not) appear in ascending order from left to right. For example, there are (obviously?) five 123-avoiding permutations of the numbers 1, 2, 3:

132 213 231 312 321

Question: How many 123-avoiding permutations are there of the numbers from 1 to n?

- A permutation of the numbers from 1 to n is **132-avoiding** if no number has a smaller number somewhere to its right, and an even smaller number somewhere to its left. For example, there are five 132-avoiding permutations of the numbers 1, 2, 3:

123 213 231 312 321

Question: How many 132-avoiding permutations are there of the numbers from 1 to n?