Metric Topology I

UW Math Circle – Advanced Group

Session 3 (10 October 2013)

Let S be a set. A *metric* d on S is a function from pairs of points in S to the real numbers. This means that for any two points a and b in S, we have a number $d(a,b)$, the distance between a and b. We also require this function to satisfy some properties:

1. Identity: $d(a,a) = 0$ for all a.
2. Positivity: $d(a,b) > 0$ for all distinct a and b.
3. Symmetry: $d(a,b) = d(b,a)$ for all a and b.
4. Triangle inequality: $d(a,b) + d(b,c) \geq d(a,c)$ for all a, b, and c.

Here are some examples in the plane. We use a Cartesian coordinate system to represent every point in the plane as a pair (x,y).

- The Euclidean metric: $d((x_1,y_1),(x_2,y_2)) = \sqrt{|x_1 - x_2|^2 + |y_1 - y_2|^2}$. This is the length of the straight line segment between (x_1,y_1) and (x_2,y_2). This is an example of a metric for which the triangle inequality is difficult to verify.
- The Manhattan metric: $d((x_1,y_1),(x_2,y_2)) = |x_1 - x_2| + |y_1 - y_2|$. This is length of a path which first goes north or south along "avenues" until reaching the right "street", then west or east along a "street". This explains the name.
- The "pessimist's metric": $d((x_1,y_1),(x_2,y_2)) = \max(|x_1 - x_2|, |y_1 - y_2|)$. Its more offical name is the Chebyshev metric. If our points are coordinates on a chessboard, this is the smallest number of moves it takes for a king to get from (x_1,y_1) to (x_2,y_2).
- The "optimist's metric" $d((x_1,y_1),(x_2,y_2)) = \min(|x_1 - x_2|, |y_1 - y_2|)$ is not a metric. It does not satisfy positivity because $d((0,0),(0,1)) = 0$. It also does not satisfy the triangle inequality because $d((0,1),(0,0)) + d((0,0),(1,0)) = 0 + 0 = 0$, but $d((0,1),(1,0)) = 1$.

The Euclidean metric, Manhattan metric, and pessimist's metric can all be generalized to three or more dimensions. Additionally, they are translation-invariant. This means that the distance between two points does not change when both points are shifted in any direction.

They Manhattan and Euclidean metrics are special cases of the metric d_p, where $p \geq 1$, defined by $d_p((x_1,y_1),(x_2,y_2)) = \sqrt[p]{|x_1 - x_2|^p + |y_1 - y_2|^p}$. The case $p = 1$ is the Manhattan metric. The case $p = 2$ is the Euclidean metric. As p grows to $+\infty$, d_p approaches the Chebyshev metric.

A *circle* is the set of points at a given distance (the *radius*) from a given point (the *center*). We can draw circles of radius 1 and center $(0,0)$ in several metrics (see the figure).

A *disc* is the area inside a circle – the set of points at a distance less than a given one. Don't confuse the two!

We now consider a set S in the plane. We write S^c for the complement of S (the set of points not in S). We made the following definitions:

- A point x is an *interior point* of S if there is a small disc around x which is entirely inside S.
- A point x is an *exterior point* of S if there is a small disc around x which is entirely outside S.
- A point x is a *boundary point* of S if it is not an interior point or an exterior point.
- S is open if it contains none of its boundary points.
- S is closed if it contains all of its boundary points.

When we say "disc", we could use the Manhattan metric, Euclidean metric, Chebyshev metric – it does not matter. This is because a disc in any metric contains a smaller disc in any other metric.

We also proved the following:

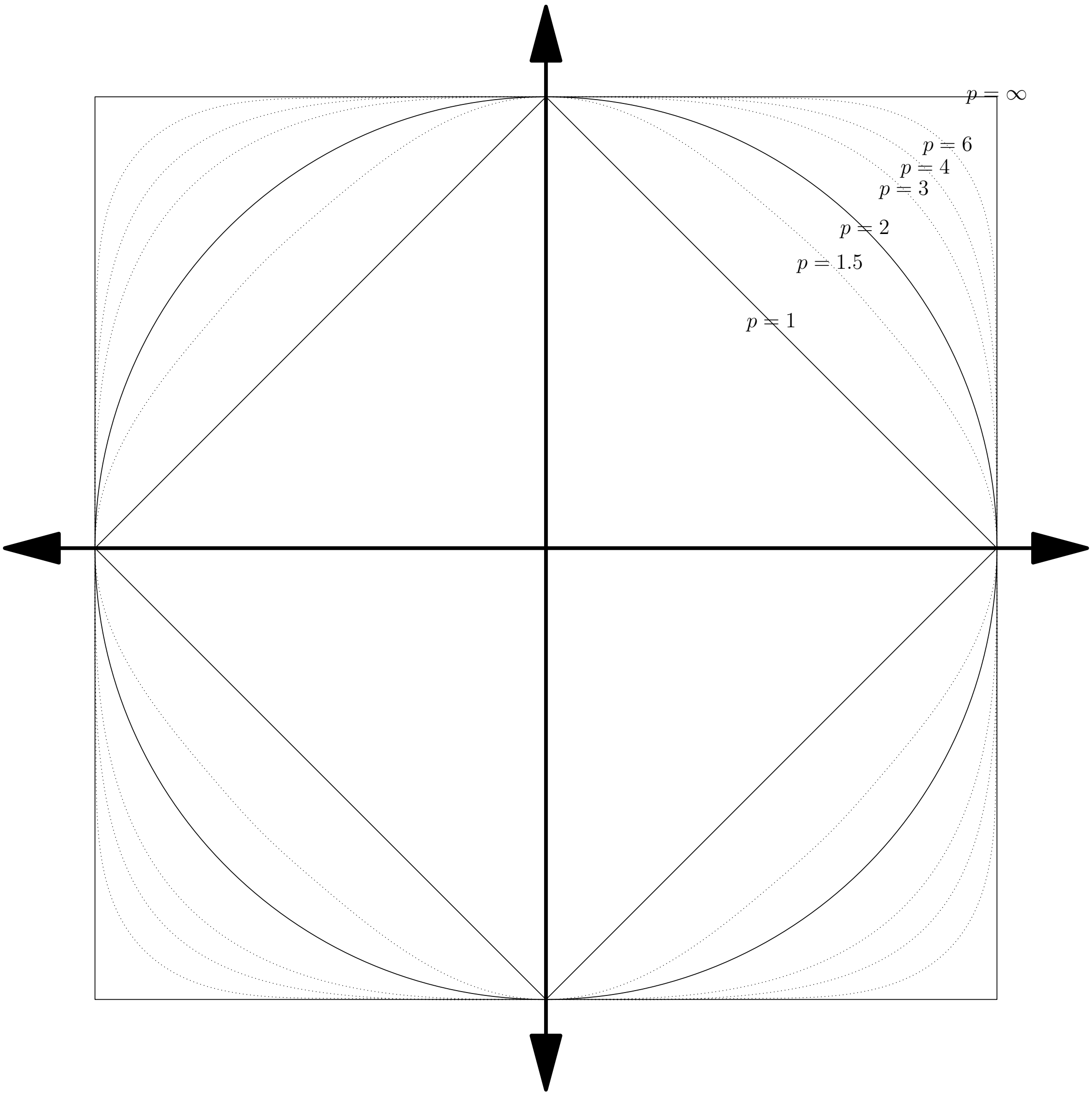
- If x is an interior point of S, then $x \in S$.

  If x is an exterior point of S, then $x \in S^c$.
- The exterior of S is the interior of S^c.

  The interior of S is the exterior of S^c.

  The boundary of S and the boundary of S^c are the same.
- S is open if and only if S^c is closed.

  S is closed if and only if S^c is open.


$p=\infty$
$p=6$
$p=4$
$p=3$
$p=2$
$p=1.5$
$p=1$