Group Theory III

UW Math Circle – Advanced Group

Session 16 (7 February 2014)

A *permutation* of $\{1, 2, \ldots, n\}$ is a bijection from $\{1, 2, \ldots, n\}$ to itself. That is, it is a reordering of the numbers from 1 to n.

We can write a certain permutation of 8 elements as follows: $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 8 & 6 & 3 & 5 & 2 & 1 & 4 \end{pmatrix}$. This means: 1 goes to 7, 2 goes to 8, 3 goes to 6, etc. Sometimes we drop the upper line and just write $\sigma = [78635214]$.

We define multiplication of permutations ab to mean "first do a, then do b".[1] For example, if σ is as above and $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 7 & 1 & 4 & 2 & 3 & 8 & 5 \end{pmatrix}$, then we can compute $\sigma\tau$ as follows. 1 goes to 7, which goes to 8; 2 goes to 8, which goes to 5; etc. We find $\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 5 & 3 & 1 & 2 & 7 & 5 & 4 \end{pmatrix}$.

Theorem 7. *The set of permutations of n elements forms a group, denoted S_n or $\mathfrak{S}_n$, and $|\mathfrak{S}_n| = n!$. This group is called the* symmetric group *on n elements.*

A permutation can be decomposed into cycles, for example, $w = (1\ 7)(2\ 8\ 4\ 3\ 6)(5)$. This means "1 goes to 7, 7 goes to 1; 2 goes to 8, 8 goes to 4, 4 goes to 3, 4 goes to 6, 6 goes to 2; 5 goes to itself".

Theorem 8. *The order of an element $w \in \mathfrak{S}_n$ is the least common multiple of the lengths of the cycles of w.*

A *transposition* is a permutation which reverses two elements and keeps all other elements fixed. An *even permutation* is one which can be written as a product of an even number of transpositions. A permutation that is not even is *odd*.

Theorem 9. *The set of even permutations of n elements forms a subgroup of $\mathfrak{S}_n$, denoted A_n, the* alternating group, *and $|A_n| = \frac{1}{2}|\mathfrak{S}_n| = \frac{n!}{2}$.*

[1] In some other places you may see it written the other way: ab means first b, then a.