Start-of-the-Year Olympiad

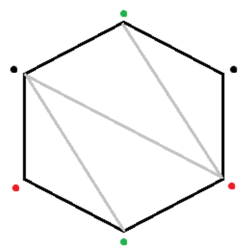
UW Math Circle – Advanced Group

Session 1 (26 September 2013)

1. Is it possible to place sixteen non-zero numbers in a 4×4 square in such a way that the sum of the numbers in the corners of any 2×2, 3×3 or 4×4 square is 0?

2. Prove that there do not exist odd integers a, b, c, d, e, f such that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}+\frac{1}{f} = 1$.

3. Each vertex of a regular n-gon is coloured green, red or black so that no two adjacent vertices are the same colour and there is at least one vertex of each colour. Prove that it is possible to cut this n-gon into triangles with several diagonals such that each triangle has one vertex of each colour.



4. A city has the shape of a 10×10 square grid (the grid lines are streets and the squares are city blocks). What is the greatest number of street segments (sections of streets along single blocks) that can be closed for repairs so that each intersection can still be reached from any other?

5. The sum of three integers x, y and z is divisible by 30. Prove that $x^5 + y^5 + z^5$ is also divisible by 30.

6. The Fibonacci numbers are defined as follows: $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n = 0, 1, 2, \ldots$. Prove that there is some nonzero Fibonacci number which is divisible by 2013.

7. There are 99 benches inside a city park.

   (a) The FBI placed a hidden microphone under each bench. Then the NSA placed a hidden microphone exactly in the middle between each pair of FBI microphones. Prove that there are now at least 294 microphones in the park.

   (b) A secret agent sits on each bench and spies on the secret agent on the nearest bench. It is known that all the distances between pairs of benches are different. Prove that at least one secret agent is not being spied on.

   (c) Prove that no secret agent is being spied on by more than five others. (For this problem, assume again that all the distances between benches are different.)