Problem Set 6 Solutions

UW Math Circle – Advanced Group

Session 9 (21 November 2013)

1. (a) We must find a sequence of steps to get from any rational number to 0 by reversing the operations $x \mapsto x+1$, $x \mapsto -x$, and $x \mapsto \frac{1}{x}$, that is, using $x \mapsto x-1$, $x \mapsto -x$, and $x \mapsto \frac{1}{x}$. Here is an algorithm to do this:

   i. If the number is negative, do $x \mapsto -x$.
   ii. While $x \geq 1$, do $x \mapsto x-1$. (After this step, $0 \leq x < 1$.)
   iii. If $x = 0$, done.
   iv. Do $x \mapsto \frac{1}{x}$.
   v. Go to ii.

   For example: $-\frac{5}{7} \mapsto \frac{5}{7} \mapsto \frac{7}{5} \mapsto \frac{2}{5} \mapsto \frac{5}{2} \mapsto \frac{3}{2} \mapsto \frac{1}{2} \mapsto 2 \mapsto 1 \mapsto 0$.
   Let's prove that this algorithm always terminates. Since after step ii the numerator is strictly smaller than the denominator and iv inverts the numerator with the denominator, steps ii-iv strictly decrease the denonimator. So, eventually the denominator will decrease to 1, after which ii results in 0 and iii tells us to end.

   (b) This time we can only use the operations $x \mapsto x-1$ and $x \mapsto \frac{-1}{x}$. Here is a modification to the algorithm that will eventually give us 0:

   i. If the number is negative, do $x \mapsto \frac{-1}{x}$.
   ii. While $x > 0$, do $x \mapsto x-1$. (After this step, $-1 < x \leq 0$.)
   iii. If $x = 0$, done.
   iv. Go to i.

   For example: $-\frac{5}{7} \mapsto \frac{7}{5} \mapsto \frac{2}{5} \mapsto -\frac{3}{5} \mapsto \frac{5}{3} \mapsto \frac{2}{3} \mapsto -\frac{1}{3} \mapsto 3 \mapsto 2 \mapsto 1 \mapsto 0$.
   As above, the denominator decreases, so the algorithm will terminate.

2. Suppose no such m exists. Then, for any small distance $(\frac{1}{n})$, there are points $x \in C$ and $y \in D$ such that $d(x,y) < \frac{1}{n}$. Take a sequence of points (x_n) in C such that x_n is less than $\frac{1}{n}$ from some point of D. Because D is bounded, some subsequence of x_n converges to a boundary point of D, but, because C is compact, this subsequence must converge to a point of C. That is, some boundary point of D is in C. D contains all of its boundary points, so C and D intersect. But D and C were assumed to be disjoint, contradiction.

3. (a) Take any equilateral triangle with side length 1. Two of the vertices are the same color – there is our segment.

   (b) Take a hexagon $ABCDEF$ and let its center be O. Suppose there is no equilateral triangle with all vertices the same color.

Without loss of generality, O is green.

If all of A, B, C, D, E, F are blue, then ACE is an all-blue equilateral triangle. So, at least one of them (without loss of generality, A) is green.

Then B is blue (lest OAB be an all-green triangle), and so is F (from OAF). D is green (from FBD) and E and C are blue (from ODE and ODC).

Let X be the intersection point of lines AF and DE. X must be green (from FEX), but it must also be blue (from ADX). Contradiction. So, there exists an all-green or all-blue equilateral triangle.