Problem Set 5 Solutions

UW Math Circle – Advanced Group

Session 8 (14 November 2013)

1. To disprove this, consider the sequence $0, 1, 0, 2, 0, 3, 0, 4, \ldots$. Clearly the only accumulation point is 0. If this sequence converged, it would converge to the only accumulation point, but, in fact, for *any* disc around 0 there are infinitely many terms outside the disc.

   If we considered ∞ as a possible accumulation point, this would be true: the plane together with ∞ is compact, so, by the Bolzano-Weierstraß theorem, any sequence has a convergent subsequence. The point to which this convergent subsequence converges would be an accumulation point. The sequence given above would not be a counterexample because it would have two accumulation points: 0 and ∞.

2. This is really a trick question: a sequence has an accumulation point at x if and only if there is has a subsequence that converges to x. (Prove it!) So, this is exactly the usual Bolzano-Weierstraß theorem.

3. Let's show that there must be at least 21 flights. Indeed, among any two airlines there must be at least 14 flights total, so together there must be at least $\frac{14\cdot 3}{2}$ flights. (Another way to see this is that if there are 20 flights, then some airline has at least 7 flights, and removing it would leave only 13; in this case, you must also make such an argument for $19, 18, \ldots$.)

   It is not difficult to find a way to do this with 21 flights.

4. Let $a = f(1)$, $b = f(2)$, $c = f(3)$, $d = f(4)$, $e = f(5)$. We have the following chains:

$$(a - 5 \mapsto) 1 \mapsto a \mapsto 6 \mapsto a + 5 \mapsto 11 \mapsto \ldots$$

$$(b - 5 \mapsto) 2 \mapsto b \mapsto 7 \mapsto b + 5 \mapsto 12 \mapsto \ldots$$

$$(c - 5 \mapsto) 3 \mapsto c \mapsto 8 \mapsto c + 5 \mapsto 13 \mapsto \ldots$$

$$(d - 5 \mapsto) 4 \mapsto d \mapsto 9 \mapsto d + 5 \mapsto 14 \mapsto \ldots$$

$$(e - 5 \mapsto) 5 \mapsto e \mapsto 10 \mapsto e + 5 \mapsto 15 \mapsto \ldots$$

   The first (parenthesized) item appears if the value is positive.

   We see that if $x \equiv y \pmod 5$, then $f(x) \equiv f(y) \pmod 5$. That is, f takes remainders modulo 5 to remainders modulo 5. We also see that if $f(x) \equiv y \pmod 5$, then $f(y) \equiv x \pmod 5$ and that we could not have $x \equiv f(x) \pmod 5$ for any x. So, the remainder classes modulo 5 can be split into pairs (x, y), where $f(x) \equiv y \pmod 5$ and $f(y) \equiv x \pmod 5$. But there are 5 such remainder classes, contradiction.

If 5 were replaced with 6, there would be no problem. For example, we could have $f(x) = x+3$, or, more interestingly, something similar to

$$f(x) = \begin{cases} x+1 & x \equiv 0 \pmod 6 \\ x+5 & x \equiv 1 \pmod 6 \\ x+2 & x \equiv 2 \pmod 6 \\ x+2 & x \equiv 3 \pmod 6 \\ x+4 & x \equiv 4 \pmod 6 \\ x+4 & x \equiv 5 \pmod 6 \end{cases}.$$

Then we would have the pairs $0 \leftrightarrow 1$, $2 \leftrightarrow 4$, $3 \leftrightarrow 5$.