Problem Set 2 Solutions

UW Math Circle – Advanced Group

Session 4 (17 October 2013)

1. Suppose A and B are open, so for any point $x \in A$ there is a small disc around x contained in A and for any point $x \in B$ there is a small disc around x contained in B.

   Consider a point $x \in A \cup B$. x is either in A or in B (or both). If $x \in A$, take a small disc around x contained in A. This disc is also in $A \cup B$. If $x \in B$, take a small disc around x contained in B. This disc is also in $A \cup B$. So, for any $x \in A \cup B$, there is a small disc around x contained in $A \cup B$. $A \cup B$ is open.

   Consider a point $x \in A \cap B$. Let D_A be a small disc around x contained in A. Let D_B be a small disc around x contained in B. Let D be the intersection of D_A and D_B. D is also a disc around x, and it is contained in $A \cap B$. So, for any $x \in A \cap B$, there is a small disc around x contained in $A \cap B$. $A \cap B$ is open.

2. First, notice that if a set is open and closed, it both contains all of its boundary points and contains none of its boundary points. So, it must have no boundary points.

   Suppose S has no boundary points. Then every point of the plane is either an interior point or S or an exterior point of S.

   If there are two points x and y such that x is an interior point or S and y is an exterior point of S, consider the line segment from x to y. Let p be the point on this segment closest to x with the property that all points past p on this segment are exterior points.[1] p could not be an interior point because there are only exterior points on one side of it. p could not be an exterior point because then all points in a small disc around p are exterior, and we could move p to a point within this disc closer to x and still have all points past it be interior points, contradicting that p was the closest point to x with this property. So, p is a boundary point, contradiction.

   It follows that either all points of the plane are interior points of S (so S is the whole plane) or all points of the plane are exterior points of S (so S is the empty set).

3. It is easy to show that within any disc in the plane there are both points with both rational coordinates and points with at least one irrational coordinate. So, if x is any point in the plane, any small disc around x contains points in S and points outside of S, so x is a boundary point. Thus, the interior and exterior of S are both empty. The boundary of S is the whole plane.

[1] Here we have used the completeness property of the real numbers: any set of points on a segment has a "leftmost right bound" and a "rightmost left bound".

4. **Solution 1** (found by Thomas): Denote the two opinions by T (torus) and F (flat). First we show that no wise man could change his opinion at every turn. It is easy to show that if someone changes his opinion at every turn, then so do both of his neighbors, and both of his neighbors must disagree with him. It follows that on the first turn, the opinions alternated around the circle: ...FTFTFTF.... But there is an odd number of wise men, contradiction.

   Now observe that if someone does not change his opinion on a given turn, then neither did at least one of his neighbors, and he and his neighbor will never change their opinions again. So, once someone has stopped changing his opnion, he stops forever.

   Because no wise man could change his opinion at every turn, for every wise man there is a turn where he will not change his opinion. After this turn he will never change his opinion again.

   So, all opinion-changing will cease after a finite number of turns.

   Solution 2 with semi-invariants (found by Benjamin and Andrew): Let us count the number of pairs of wise men who are neighbors and disagree with each other. It is easy to see that at every turn, this number either decreases or stays the same: two neighbors who used to agree could not begin to disagree. But if this number stays the same for infinitely many turns, then we see as in the first solution that the opinions must be alternating, which is impossible. Because our number could never become negative, this process will eventually terminate.