Problem Set 10 Solutions

UW Math Circle – Advanced Group

Session 16 (6 February 2014)

1. See the official solution, `http://www.bamo.org/attachments/bamo2005examsol.pdf`.

2. (a) Obviously, if A and B are sets in the plane, then $A \triangle B$ is also a set in the plane.
   Associativity: You can convince yourself that $(A \triangle B) \triangle C = A \triangle (B \triangle C)$ by drawing a diagram.
   Identity: The empty set.
   Inverses: Every set is its own inverse.
   (b) $\triangle$ is not an operation on H, that is, if A and B are open sets, then $A \triangle B$ need not be open. For example, take A to be the open disc of radius 2 centered at 0 and B to be the open disc of radius 1 centered at 0. $A \triangle B$ is an annulus that includes its inner boundary but not its outer boundary (so it is not open).

3. (a) Equivalently, we are to find the subgroup of $\mathbb{Z}$ generated by 123, 405, and 321. It is generated by one element, the gcf of those three numbers, which is 3. So, all floors with number divisible by 3 can be reached.
   (b) Suppose this is possible, so there exist buttons $\frac{m_1}{n_1}, \frac{m_2}{n_2}, \ldots, \frac{m_k}{n_k}$ such that every rational number can be reached with them. But let $n = \operatorname{lcm}(n_1, n_2, \ldots, n_k)$. Every floor that can be reached could be written as a fraction with denominator n. However, we cannot reach, for example, floor $\frac{1}{n+1}$. Contradiction. So, this is impossible.