Problem Set 7 Solutions

UW Math Circle – Advanced Group

Session 11 (12 December 2013)

1. We give an algorithm for writing any number as a sum of distinct primes or 1. (We could also show this by strong induction.)

   Begin with a number n. If n is prime, we are done. Else, take a prime p between $\frac{n}{2}$ and n and write it down. Subtract p from n.

   The remainder after subtracting p from n is less than $\frac{n}{2}$. Repeat until getting a prime number – then write the prime number down.

   All the numbers you have written down will be distinct and add up to n.

   For example, begin with 20. Take $p = 11$ (write down 11).

   We are left with 9. Take $p = 5$ (write down 5).

   We are left with 4. Take $p = 3$ (write down 3).

   We are left with 1, which is prime or 1 (write down 1).

   Get $20 = 11 + 5 + 3 + 1$.

2. (a) It is obvious that this is not prime for $n = 41$. It is also true, though less obvious, that it is not prime for $n = 40$.

   (b) Suppose p is such a polynomial. Let $q = p(0) = c_0$, so q is prime. Now, $p(q)$ is divisible by q and is prime, so $p(q) = q$. Similarly, $p(2q) = q$, $p(3q) = q$, and so on. That is, p takes the value q infinitely many times ($p(x) - q$ has infinitely many roots), so p must be constant.

3. Any center of a disc of diameter 1 fitting inside a 3×3 squares must lie inside (or on the boundary) of a 2×2 square exactly in the middle of the 3×3 square. Divide this 2×2 squares into nine $\frac{2}{3} \times \frac{2}{3}$ squares. The largest distance between two points in one of these small squares is $\frac{2}{3}\sqrt{2} < 1$, so no more than one circle can have its center inside or on the boundary of the small squares. So there could not be more than nine discs.