Start-of-the-Year Olympiad Solutions

UW Math Circle – Advanced Group

Session 2 (3 October 2013)

1. Yes, for example:

1	-1	-1	1
-1	1	1	-1
1	-1	-1	1
-1	1	1	-1

2. Suppose such odd numbers did exist, so

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}+\frac{1}{f}=\frac{bcdef+acdef+abdef+abcef+abcdf+abcde}{abcdef}=1.$$

Each term in the numerator is odd, so the numerator is even. The denominator is odd. Thus the value of the fraction cannot be 1 (an odd integer).

3. First, notice that this is trivial if there is only one vertex of some colour. So, we may suppose there are at least two vertices of each colour.

Otherwise, by induction on n, the number of vertices. For $n = 3$, a triangle, this is trivial.

Suppose that any n-gon with vertices coloured in three colours with no adjacent vertices the same colour can be cut into red-green-black triangles.

Consider now an $n + 1$-gon with vertices coloured in three colours with no adjacent vertices the same colour. We can find three consecutive vertices of different colours. We "cut off" this triangle with a diagonal, leaving an n-gon in which there are still no adjacent vertices of the same colour and there is at least one vertex of each colour (remember that we assumed there were 2 of each colour before cutting off). Thus we have reduced to the case of n.

4. Draw a graph corresponding to the city grid in which vertices are intersections and edges are city blocks. This graph has 121 vertices and 220 edges. Suppose we have closed the greatest possible number of edges to keep the graph connected (i.e., you can reach any intersection from any other). The resulting graph should have no cycles – it should be a tree.

A tree with 121 vertices has 120 edges. Now, in the original graph, there are 220 edges, so 100 street segments can be closed.

(For example, we could close all of the north-south street segments except those along the west side.)

5. It is easy to check that for all n,

$$\begin{aligned} n^5 &\equiv n \pmod{2}, \\ n^5 &\equiv n \pmod{3}, \\ n^5 &\equiv n \pmod{5}. \end{aligned}$$

Therefore, $x^5 + y^5 + z^5 \equiv x + y + z$ modulo 2, 3, and 5. Because $x + y + z$ is divisible by 2, 3, and 5, so is $x^5 + y^5 + z^5$ – this quantity is also divisible by $\text{lcm}(2, 3, 5) = 30$.

6. For each pair of consecutive Fibonacci numbers (F_k, F_{k+1}) look at the remainders modulo 2013. There is a finite number – 2013^2, in fact – of possible pairs of remainders, so by the pigeonhole principle there are two different indices m and n (suppose $m < n$) such that

$$\begin{aligned} F_n &\equiv F_m \pmod{2013}, \\ F_{n+1} &\equiv F_{m+1} \pmod{2013}. \end{aligned}$$

But then

$$\begin{aligned} F_{n-1} &\equiv F_{m-1} \pmod{2013}, \\ F_{n-2} &\equiv F_{m-2} \pmod{2013}, \end{aligned}$$

and so on until $F_{n-m} \equiv F(m - m) = F_0 = 0$. Thus we have found a Fibonacci number divisible by 2013.

7. (a) We have to show that the NSA placed at least 195 microphones. Take the two benches that are farthest apart, call them A and B. The midpoints of A with the other 97 benches are all inside a circle with center A and radius $\frac{1}{2}|AB|$. Similarly, the midpoints of B with the other 97 benches are all inside a circle with center B and radius $\frac{1}{2}|AB|$. Those two circles do not intersect, so we have already found $97 + 97 = 194$ microphones. Finally, there is a microphone at the midpoint of A and B.

   (b) Suppose to the contrary that every bench is being watched.

   Observe that if more than one person is watching some bench, then there exists a bench that is not being watched. So, if every bench is being watched, then every bench is being watched by one person.

   Take the two benches that are closest together; the agents on them must be watching each other, and nobody else can be watching either of those two agents.

Repeat with the remaining 95 agents; eventually we have broken the agents into pairs and are left with one extra agent (contradiction).

(If we had an even number of benches, it would be possible that all benches are being watched. It's easy to find an example.)

(c) Claim: If a bench A is being watched from benches B and C, then $\angle BAC \geq 60°$. In particular, if $\angle BAC = 60°$, then $|AB| = |AC|$.

Proof: We have $|BC| < |AC|$ and $|BC| < |AB|$. Thus $|BC|$ is the shortest side of $\triangle ABC$ and so the angle opposite it, $\angle BAC$, is the smallest angle and is no greater than $\frac{180°}{3} = 60°$. It is easy to see how the second part of the claim follows.

Now, if six or more people are watching a bench A, then either some angle between two of them is less than $60°$, contradiction, or they are all at angles of $60°$. But then all distances from A to the benches watching A are the same, contradicing the initial assumption.