Problem Set 5

UW Math Circle – Advanced Group

Session 7 (7 November 2013)

Q: What do you call the irrational fear of convergent sequences?
A: Zenophobia.

1. Prove or disprove: If a sequence of points in the plane has exactly one accumulation point, then the sequence converges to this accumulation point.

2. (a) Show that any compact set S contains all accumulation points of sequences in S.
   (b) (Another form of Bolzano-Weierstraß theorem) Show that a set S in the plane is compact if and only if any sequence in S has an accumulation point in S. *(Hint: Use the Bolzano-Weierstraß theorem and (a).)*

3. In a certain country there are 15 cities. Some of the cities are connected by air routes, each operated by one of three airlines. It is possible to travel from any city to any other by air, possibly with transfers between airlines. It turns out that if any one of the companies goes bankrupt, it will still be possible to get from any city to any other. What is the smallest possible number of air routes there could be in the country?

4. (IMO 1987) Let $\mathbb{N}$ denote the set of positive integers. Does there exist a function $f : \mathbb{N} \to \mathbb{N}$ such that $f(f(n)) = n + 5$ for every n?