# Problem Set 20

UW Math Circle – Advanced Group

Session 28 (22 May 2014)

1. (a) (MHO 2013) Suppose that you have a triangle in which all three side lengths and all three heights are distinct integers. Prove that among these six lengths there cannot be four prime numbers.

   (b) (USSR 1975) Two altitudes of a triangle have lengths 10 and 6. Prove that the third is not longer than 15.

2. The area of the triangle ABC shown in the figure is 1. The points D, E, and F are trisection points of the sides BC, CA, and AB. The points G, H, and I are the intersection points of AD, BE, and CF. What is the area of the small triangle GHI?[1]
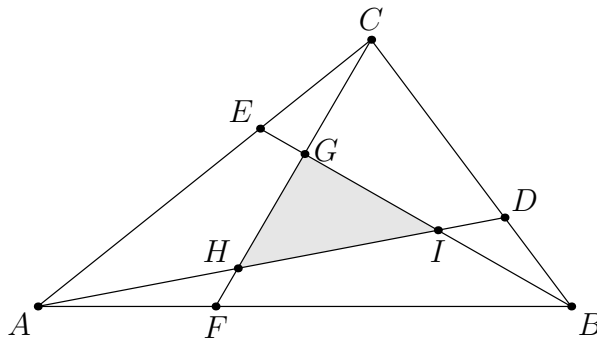
3. (MHO 2012) Ann and Bob play a game with 40 stones on a table. On each turn, a player can select any two piles of stones and combine them into one bigger pile, as long as this new pile would not contain more than the half of all the stones on the table. A player who can't make a move loses. Ann goes first. Who wins?

4. (ibidem; Russia 2000) Katniss is thinking of a positive number less than 100 (call it x). Peeta is allowed to pick any positive integers N and M, both less than 100, and Katniss will give him the greatest common divisor of $x + M$ and N. Peeta can do this up to seven times, after which he must name Katniss's number x, or he will die. Can Peeta ensure his survival? (Bonus: Do it in 6 steps! Extra bonus: Show that you can do it in **3** steps!!)

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[1]Bonus: Prove *Routh's theorem*, which states that if D, E, and F divide the sides in ratios of $1 : x$, $1 : y$, and $1 : z$, then the area of GHI is

$$\frac{(xyz - 1)^2}{(xy + y + 1)(yz + z + 1)(zx + x + 1)}.$$

If $x = y = z$, you can check that this simplifies to

$$\frac{(x - 1)^3}{x^3 - 1}.$$

In the problem above, we have $x = y = z = 2$.