Problem Set 12

UW Math Circle – Advanced Group

Session 18 (20 February 2014)

1. Consider the curve $y^2 = x^3 - x + 1$.
   (a) Graph it.
   (b) Find three points on this curve with integral coordinates, call them P, Q, R.
   (c) "Verify" that $(P + Q) + R = P + (Q + R)$.
   (d) Can you use these three integral points to find any more?

2. In class, we did an example with square pyramids and squares. Let's consider a similar problem with triangular pyramids. What sizes of triangular pyramids can be reshaped into squares?
   (a) Our triangular pyramids will have 1 block in the first layer, 3 in the second layer, 6 in the third layer, etc. Show, by induction, that the number of blocks in an x-level triangular pyramid is
   $$1 + 3 + 6 + 10 + \cdots + \frac{x(x+1)}{2} = \frac{x(x+1)(x+2)}{6}.$$
   Bonus: Find a combinatorial proof of this. Notice that this is really
   $$\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{x+1}{2} = \binom{x+2}{3}.$$
   (b) By solving this problem for possible x and letting the side length of the square be y, we should have a cubic
   $$y^2 = \frac{x(x+1)(x+2)}{6}.$$
   We showed that, even though this isn't quite the form of an elliptic curve, we can always transform it into one by a substitution.
   (c) $(0, 0)$ and $(1, 1)$ are solutions. Can you find any others?