Problem Set 11

UW Math Circle – Advanced Group

Session 16 (6 February 2014)

1. (BAMO 2008) Call a year *ultra-even* if all of its digits are even. Thus, 2000, 2002, 2004, 2006, and 2008 are all ultra-even years.
   (a) In the years between the years 1 and 10000, what is the longest possible gap between two ultra-even years? Give an example of two ultra-even years that far apart with no ultra-even years between them.
   (b) What is the second-shortest possible gap?

   (To be written down, as last week.)

2. Suppose $\sigma \in \mathfrak{S}_n$. Prove that the order of σ divides the order of $\mathfrak{S}_n$.

3. A fleet of 100 starships of different sizes approaches a Martian spaceport in a line. Martian customs and diplomatic rules require that the largest starship land first, the second-largest starship second, . . . , and the smallest starship last.

   Landing conditions are very difficult. Only two manœuvres are possible:
   - Manœuvre A: the front two starships in the line switch places.
   - Manœuvre B: the last starship in the line moves to the front.

   Prove that the starships can get in the right order by a combination of these moves. (You've just shown that $\mathfrak{S}_n$ is generated by $(1\ 2)$ and $(2\ 3\ 4\ \ldots\ n\ 1)$.)

4. (Bubble sort is slow) Today we showed that every permutation in $\mathfrak{S}_n$ is a product of transpositions of adjacent elements. Prove that any permutation is a product of *no more than* $\binom{n}{2} = \frac{n(n-1)}{2}$ transpositions of adjacent elements.

   Bonus: Find a permutation that cannot be written as a product of fewer than $\binom{n}{2}$ adjacent transpositions, that is, show that we cannot replace $\binom{n}{2}$ by anything smaller in the above problem.