Problem Set 10

UW Math Circle – Advanced Group

Session 15 (30 January 2014)

1. (BAMO 2005) There are 1000 cities in the country of Euleria, and some pairs of cities are linked by dirt roads. It is possible to get from any city to any other city by traveling along these roads. Prove that the government of Euleria may pave some of the roads so that every city will have an odd number of paved roads leading out of it.

   Like last week, write down your solution.[1]

2. (a) Let $G = \{$sets of points in the plane$\}$ with the operation $\triangle$ defined by
   $$A \triangle B = (A \cup B) \setminus (A \cap B).$$
   Prove that G forms a group with the operation $\triangle$.

   (b) Now let $H = \{$open sets of points in the plane$\}$. Is H a subgroup of G?

3. (a) The French billionaire Marie-Jeanne de l'Escargot built the infinite Hôtel Bézout, with one floor for each integer $n \in \mathbb{Z}$. The entrance is on floor 0. The elevator has three buttons – MOVE BY 123, MOVE BY 405, and MOVE BY 321 – and a switch with two positions – GO UP and GO DOWN.[2] Which floors of the hotel can be reached by the elevator?

   (b) Jean-Marie-François-Maximilien Galois, her competitor, wants to build the infinite Hôtel des Groupes, with one floor for each rational number. Show that he cannot design a similar elevator with a finite number of buttons. (You've just shown the group of rational numbers with the operation of addition is not finitely generated.)

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[1]If you are *really* stuck, read the first few lines of one of the solutions at http://www.bamo.org/attachments/bamo2005examsol.pdf for a hint.

[2]When you press a button, you go up or down (depending on the position of the switch) by the number written on the button you pressed.