Math Circle - Pigeonhole Principle

The Pigeonhole Principle. If we have n pigeonholes and at least $n + 1$ pigeons are placed in them, then at least one pigeonhole will contain *more than one* pigeon.

Proof. If every pigeonhole contained no more than one pigeon, then there would be no more than n pigeons total. □

Notice that the Pigeonhole Principle does not tell us which of the pigeonholes will be extra crowded, just that at least one of them is guaranteed to be.

The hardest part about pigeonhole problems is figuring out what are the pigeons, and what are the pigeonholes. Some easy examples:

1. Prove that among any 5 cards in a deck, there are at least two of the same suit.

2. Prove that among any 13 people, at least two of them were born in the same month.

3. How many people would you need before you can guarantee that at least two share the same birthday?

The Generalized Pigeonhole Principle. If we have n pigeonholes and at least $nk + 1$ pigeons are placed in them, then at least one pigeonhole will contain *more than* $\_\_k\_\_$ pigeons.

Proof. If every pigeonhole contained no more than k pigeons, then there would be no more than nk pigeons total. □

The Generalized Pigeonhole Principle is no more difficult than the regular Pigeonhole Principle. Let's see it used in an example.

4. A certain neighborhood in Seattle has 101 houses. Each house has between 1 and 20 windows. Show that there are 6 houses which have the exact same number of windows.

For the following problems, the pigeonholes and pigeons are not as easy to find.

5. Seven darts are thrown at a circular dart board of radius 1 foot. Prove that there must be at least two darts no more than a distance 1 foot away from each other.

6. Prove that if I put 33 rooks on an 8×8 chessboard, there must be at least five, no two of which can attack each other.

7. Being *friends* is a symmetric relationship. Prove that among a group of any $n \geq 2$ friends, there must be at least two with the exact same number of friends in the group.