Math Circle - Impossible Locks

The National Security Agency (NSA) is developing new locks to put on its internal doors in order to keep out wandering spies. Each lock consists of a sequence of keys in an initial configuration (**locked**). The keys can be switched into different modes by a small number of allowed moves. There is some final configuration of the keys which makes the lock **unlocked**.

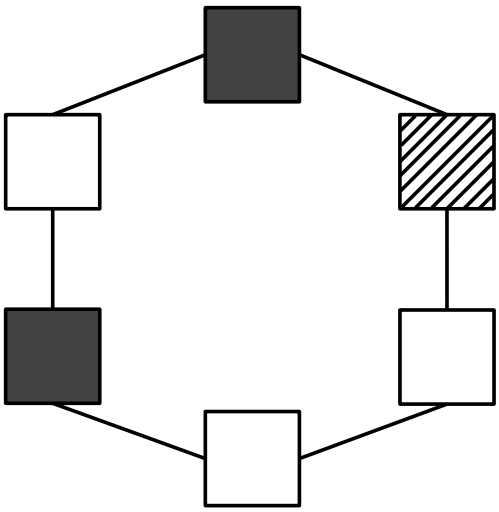
You are part of an elite team of CIA operatives whose job it is to test to make sure each lock is secure. Embarrassingly, it appears that the NSA is not too good at their job, as every lock you come across is **impossible** to unlock! Convince your superiors that each of the following locks is impossible from its given initial state.

1. In the first lock, each key is a circle which can be filled as either black or white. The lock consists of 7 of these keys arranged in a row, all initially white. In a move, you are allowed to choose exactly 4 of the keys to switch them to the opposite color. The lock is **unlocked** if all the keys are black after some finite number of moves.

2. Each key in the second lock is a square which is filled as either solid, striped, or empty. The keys are arranged in a circular pattern and can be switched as follows:

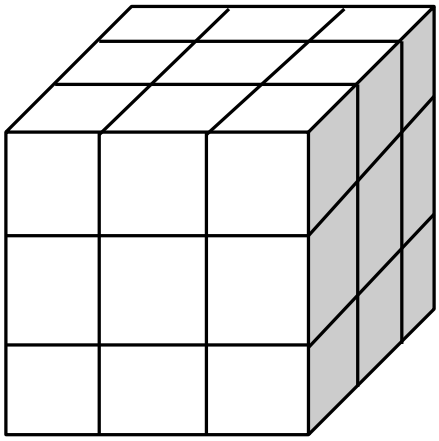
solid $\rightarrow$ striped striped $\rightarrow$ empty empty $\rightarrow$ solid

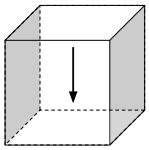
Each time you switch a key, you must also switch the two keys immediately adjacent to it. The initial configuration of the lock is shown. The lock is **unlocked** if the keys ever have all the same filling at the same time.



3. The third lock is a cube made up of 26 smaller cubes arranged into a larger cube (with no small cube at the center). Each smaller cube is a key which contains an arrow pointing in the direction of one of its neighboring keys (two of the keys are neighbors if they share a square face). You are allowed to move around the arrow in a key to point to any of the other neighboring cubes (but remember, there's no cube at the center!).

The lock is **unlocked** if you can arrange the arrows in such a way that you can start at a chosen *start key* and follow the direction of the arrows from key to key, so that you end up visiting each of the 26 keys exactly once.





4. The final lock is an 8×8 chessboard, and the keys are the 64 squares. The lock comes equipped with a new chess piece, called a **coward**. The coward starts on the bottom left corner of the board, and the lock is **unlocked** if you can move it around the board so that it lands on each square *exactly* once.

The coward can move in one of five directions: *two spaces up*, *two spaces right*, *one space diagonally up-right*, *one space down*, or *one space left*.