Combinatorics is a branch of mathematics that's all about counting things. It turns out that there are often many ways to count the same thing, and by counting in different ways you can derive some interesting equalities.

Problem: *You and five of your friends go to a restaurant. The restaurant serves three different appetizers, four different soups, five different main courses, and two different desserts. How many ways are there for you to choose an appetizer, soup, main course, and dessert? How many different ways are there for you and your friends to all choose an appetizer, soup, main course, and dessert? Why do you multiply in the first problem, but add in the second?*

Problem: *There are 20 towns in a certain country, and every pair of them is connected by an air route. How many air routes are there?*

We should take a moment to derive the equation $\sum_{n=1}^{\infty} n = \frac{n(n+1)}{2}$

Problem: *How many six-digit numbers have at least one even digit? Hint: Count the number of six-digits that have no even digits. What's the connection?*

Problem: *The Hermitian language consists of* 6 *letters. A word in this language is any combination of six letters such that some pair of which are the same. How many words are there in the Hermitian language?*

Now we move on to more advanced combinatorics...

Problem: *You have a drawer with ten socks, all of different colors. How many ways are there to choose two socks out of the ten? How many ways are there to choose three socks?*

Here, we'll spend some time talking about how/why you have to divide by the number of permutations of the socks that you select. I'll point out that all of these calculations are quite difficult and tedious, so we have a great notation for it:

Def: $\binom{n}{k}$ is the number of ways to choose k things (such as balls) from n things. In the example above, we computed $\binom{10}{2} = 45$ and $\binom{10}{3} = 120$.

Problem: *One student has* 6 *textbooks and the other has* 8*. How many ways are there for them to exchange* 3 *textbooks?*

Answer:

$$\binom{6}{3}\binom{8}{3}$$

Tell the students that this is the only kind of answer that we will accept! We don't want number solutions - we want expressions!

Theorem: $\binom{n}{k} = \binom{n}{n-k}$

Proof: Let's count the number of ways to choose k balls from n balls. On one hand, this is just $\binom{n}{k}$ by definition. On the other hand, we can also just count the number of ways to choose $n-k$ balls that we don't take. This gives $\binom{n}{n-k}$. Therefore $\binom{n}{k} = \binom{n}{n-k}$.

□

Theorem: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$

Proof: Let's count the number of ways to choose k balls from a pile of $n+1$ balls. On one hand, this is just $\binom{n+1}{k}$ by definition. On the other hand, we can choose one ball and decide whether we want to take it or not. If we don't take it, we are left with n ball that we must choose k of. If we choose to take it, we are left with n balls that we have to choose $k-1$ from. Therefore there are $\binom{n}{k} + \binom{n}{k-1}$ ways to choose k balls from $n+1$. Therefore $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

□