UW Math Circle, Spring 2013 - Homework 1

1. Fred was able to use induction to prove that all people in the world have blue eyes. He reasoned as follows:

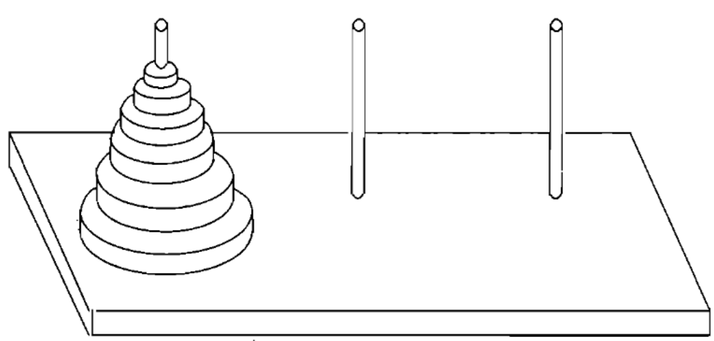
a) Base case: At least one person with blue eyes exists.

b) Inductive step: **Assume** that any collection of k people have blue eyes. Now take any collection of $k+1$ people. If we remove one person, call him Snoop Dogg, we are left with a collection of k people - which we know by hypothesis to all have blue eyes. If we put Snoop Dogg back and remove a different person, Rihanna, then we are again left with a collection of k people, so they again all have blue eyes. Therefore we have shown that everyone but Snoop Dogg has blue eyes, and everyone but Rihanna has blue eyes, meaning that all $k+1$ people have blue eyes. Thus we have shown that if k people have blue eyes, then $k+1$ people have blue eyes, so our induction is complete. Therefore all people in the world have blue eyes.

Obviously, not everyone in the world has blue eyes, so Fred's reasoning is somehow wrong. Where is Fred's mistake?

2. Peter has a game. It has three spindles on a base, with n rings on one of them. The rings are arranged in order by size (see picture). Peter is allowed to move the smallest (highest) ring on any spindle to any other spindle, as long as he does not place a larger ring on a smaller ring. Prove that

a) It is possible to move all the rings to one of the free spindles.
b) Peter can do so in $2^n - 1$ moves.
c) *Challenge:* it is not possible to do so in fewer than $2^n - 1$ moves.



3. Alex chooses $n+1$ numbers from the numbers 1, 2, 3, …, $2n$. Prove that José can always pick two of Alex's numbers such that one divides the other.

4. The country of Mathematica has 20 cities and 172 flights that connect pairs of cities. Prove that it is possible to fly from any city to any other city.