# UW Math Circle

November 1, 2012

**The Pigeonhole Principle.** If $n+1$ pigeons are placed into $n$ holes, then there will be at least one hole with more than one pigeon in it.

*Proof:* Let's pretend we can put $n+1$ pigeons in $n$ holes so that all the holes have no more than one pigeon. Then there would be no more than $n$ pigeons in total. But we have $n+1$ pigeons! That's a contradiction. So, our assumption is false. At least one of the holes has more than one pigeon. $\square$

This simple result is also known as **Dirichlet's Principle**. There is a more interesting version:

**The Generalized Pigeonhole Principle.** If $kn+1$ pigeons are placed into $n$ holes, then there will be at least one hole with more than $k$ pigeons in it.

*Proof:* Let's pretend we can put $kn+1$ pigeons in $n$ holes so that all the holes have $k$ pigeons or fewer. Then there would be no more than $kn$ pigeons in total. But we have $kn+1$ pigeons! So, there must be at least one hole with more than $k$ pigeons. $\square$

**Problems:**

1. 41 rooks are placed on a $10 \times 10$ chessboard. Show that there is a set of 5 rooks, none of whom can attack any of the others.

2. Prove that among 101 integers, you can always add some of them to get a sum that is divisible by 100.

3. 20 math circlers were passing notes in class.

   (a) If each student passed 10 notes, show that there were two students who passed notes to each other.

   (b) If each student passed 9 notes, is it still true that there were two students who passed notes to each other?

4. Ardrite scientists have launched a successful expedition to the nearby planet Gniroloc and would now like to see how their most advanced robots handle the new environment. They have constructed 64 high-tech homes in an 8*8 grid on Gniroloc and would like to send 64 robots to live there. In order for the robots not to fight, each one should have at least two robots of the same model as neighbors (to the north, south, east, or west). What is the greatest number of models that could be tested?

5. Every square of a $2013 \times 2013$ chessboard is colored one of 2012 colors. In a given row or column, if there are two squares of the same color, you can recolor the entire row/column that color. Is it always possible to make the entire chessboard one color?

6. Show that among 6 people, there are either three people who all know one another or three people, all of whom are strangers.

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Homework – Week 5

1. If you place 44 queens on a chessboard, prove that each queen attacks at least one other queen.

2. Show that for any of the 10! ways to write the numbers $1, 2, \ldots, 10$ in a row, there exist four numbers that are either in increasing or decreasing order when they are read from left to right. For example in:

$$8 \quad 6 \quad 7 \quad 5 \quad 3 \quad 10 \quad 9 \quad 4 \quad 1 \quad 2$$

the numbers $6, 5, 4, 2$ are in decreasing order when read from left to right, but there are not four numbers that are in increasing order when read from left to right. Specifically, this means: the four numbers do not have to be consecutive and it is possible to have either an increasing or decreasing list of 4 numbers but not both.

3. Prove that among 52 integers, you can always find two such that

   (a) Their difference is divisible by 51.

   (b) Their sum or difference is divisible by 100.

4. Prove that there is a number of the form 1111...1 that is divisible by 2013.

5. (a) Show that in a group of an odd number of people, there is one person who knows an even number of people and does not know an even number of people.

   (b) Show that in a group of 9 people, there are either four people who all know each other, or three people, all of whom are strangers to one another. Hint: In a group of 9 people, part (a) of this problem tells you there is a person, Belinda, who knows an even number of people and is a stranger to an even number of people. Start by showing that either Belinda knows at least 6 people or is a stranger to at least 4 people.