Math Circle - Chooses

Given nonnegative integers $n \geq k \geq 0$, recall that n *factorial* is defined as

$$n! = n(n-1)(n-2)\cdots(2)(1).$$

For convenience we define $0! = 1$. The following theorem is almost a definition.

Theorem A. *For a positive integer n, we have $n! = n \cdot (n-1)!$*

We find it convenient to define another notation, called **chooses**. Define

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

You may have seen the notation ${}_nC_k$ before. We pronounce this as n choose k, because of the following theorem.

Question 1. *Verify using this definition that $\binom{n}{k} = \binom{n}{n-k}$.*

Theorem B. *$\binom{n}{k}$ is the number of ways of choosing k things from a set of n things if order does not matter.*

Question 2. *Verify this interpretation of $\binom{n}{k}$ for $k = 0, 1,$ and n?*

Question 3. *Verify (again!) that $\binom{n}{k} = \binom{n}{n-k}$. This time use Theorem B.*

Question 4. *Is Theorem B obvious? Is it even clear that $\binom{n}{k}$ is an integer?*

Pascal's Triangle

(a) Verify the following equality:

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}.$$

(b) For a fixed nonnegative integer n, we can write down all of $\binom{n}{k}$ in a row, left-to-right, for $k = 0, 1, 2, \ldots, n-1, n$. Then using part (a), we can immediately write down the row for $n+1$ right underneath it!

Try this for $n = 4$:

$$\begin{array}{cccccccccccc} & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\ & \| & & \| & & \| & & \| & & \| & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ & & \backslash/ & & & & \backslash/ & & & & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\ \| & & \| & & \| & & \| & & \| & & \| \\ \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} \end{array}$$

(c) Start from $n = 0$ and go to $n = 7$ forming the rows described in part (b), one on top of the other. This forms a triangle, which continuing forever is called *Pascal's triangle.*

$$\begin{array}{c} 1 \\ 1 \quad 1 \\ 1 \quad 2 \quad 1 \\ 1 \quad 3 \quad 3 \quad 1 \\ 1 \quad 4 \quad 6 \quad 4 \quad 1 \\ 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\ \vdots \qquad \vdots \qquad \vdots \end{array}$$

Question 5. *How does Pascal's triangle prove that* $\binom{n}{k}$ *is always an integer?*

Question 6. *How can we use part (a) to prove Theorem B?*