Math Circle - Geometry of Equivalence Relations

Recall that a relation on a set S which is symmetric, transitive, and reflexive is called an **equivalence relation**. For equivalence relations, we'll use $\sim$ instead of $\propto$.

Definition. Let $a \in S$. Then the **equivalence class** of a is the subset $[a]$ of S

$$[a] = \{s \in S : a \sim s\}.$$

Theorem. Let $a, b \in S$. Then $[a] = [b]$ if and only if $a \sim b$.

A consequence of the previous theorem is that an equivalence relation allows you to break up the set S into disjoint groups, where each group corresponds to an *equivalence class*. This is called a **partition** of S.

For each of the following problems, let $S = \mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$. Verify that the given $\sim$ is indeed an equivalence relation (symmetric, transitive, reflexive). Then draw what the equivalence classes look like.

1. $(x, y) \sim (a, b) : y = b$.
2. $(x, y) \sim (a, b) : y - b$ is an integer.
3. $(x, y) \sim (a, b) : x^2 + y^2 = a^2 + b^2$.
4. $(x, y) \sim (a, b) : x - y = a - b$.
5. $(x, y) \sim (a, b) :$ There exists $\lambda \neq 0$ such that $\lambda(x, y) = (a, b)$.

Definition. The **quotient set** of S with respect to the equivalence relation $\sim$ is the collection of all equivalence classes:

$$S/\!\sim\; = \{[a] : a \in S\}.$$

This $S/\!\sim$ is just the formal way of grouping everything in S into their respective equivalence classes. If $a \sim b$, then $[a]$ and $[b]$ are the exact same element in $S/\!\sim$. We read this as "S mod $\sim$".

For each of the relations on $S = \mathbb{R}^2$ above, try to give a drawing for $S/\!\sim$

Important: each equivalence class you drew originally should somehow correspond to a single point in the $S/\!\sim$ picture.