Math Circle - Mathematical Induction

The principle of **mathematical induction** is an extremely useful proof technique for statements that say something about positive integers (even when the statements are not so obviously about positive integers). Any proof by mathematical induction proceeds in two steps: 1) set up a **fuse** connecting all the statements about each individual positive integer so that the truth of one statement automatically implies the truth of another and, 2) set off the fuse by introducing a **spark** which indicates the truth of one *particular* statement.

This is best seen in an example. Consider the statement

$$1 + 2 + 3 + \cdots + (n-1) + n = \frac{n(n+1)}{2} \qquad (*)$$

for all $n = 1, 2, 3, \ldots$. Even if we don't believe this statement right away, we can surely check this for any arbitrary n:

$$\begin{array}{lll} n = 1 & \rightsquigarrow 1 = \frac{1(2)}{2} & \checkmark \\ n = 2 & \rightsquigarrow 1 + 2 = \frac{2(3)}{2} & \checkmark \\ n = 3 & \rightsquigarrow 1 + 2 + 3 = \frac{3(4)}{2} & \checkmark \\ n = 8 & \rightsquigarrow 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = \frac{8(9)}{2} & \checkmark \end{array}$$

It is unsatisfying, though, that no matter how many of these we check by hand, we will never be able to explicitly check statement $(*)$ for **all** positive integers.

We are pretty good with "symbolic manipulation," though. Suppose that through some neat tricks we are able to show:

If statement $(*)$ is true for $n = k$, then it is also true for $n = k + 1$. $\qquad (\diamond)$

Consider what this would mean. We explicitly checked that $(*)$ is true for $n = 1$. Statement $(\diamond)$ then tells us that $(*)$ must also be true for $n = 2$. Now that we know statement $(*)$ for $n = 2$, statement $(\diamond)$ implies that $(*)$ is also true for $n = 3$. We've created a **fuse** of statements!

"$(*)$ for $n = 1$" $\Longrightarrow$ "$(*)$ for $n = 2$" $\Longrightarrow$ "$(*)$ for $n = 3$" $\Longrightarrow$ "$(*)$ for $n = 4$" $\Longrightarrow \cdots$

Setting off the **spark** by showing that "statement $(*)$ is true for $n = 1$" sets off the whole explosion of truth, and we immediately get that statement $(*)$ is true for all n.

We're not done yet! We still haven't done any of the work to prove statement $(\diamond)$. All we've done is show that **if** we can somehow establish the truth of statement $(\diamond)$, then this sets up the fuse. And the explicit check for $n = 1$ is the needed spark we mentioned to finish the proof of $(*)$.

We now finish the job by showing that indeed statement $(\diamond)$ is true. To show this, we need to assume that $(*)$ is true for some arbitrary $n = k$; explicitly this means

$$1 + 2 + 3 + \cdots + (k-1) + k = \frac{k(k+1)}{2}. \qquad (\square)$$

We want to somehow use this to establish the truth of $(*)$ for $n = k + 1$. That is, we want to show then that

$$1 + 2 + 3 + \cdots + k + (k+1) = \frac{(k+1)((k+1)+1)}{2}$$

This is no more than the following string of implications:

$$\begin{aligned} 1 + 2 + 3 + \cdots + k + (k+1) &= (1 + 2 + 3 + \cdots + k) + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) \qquad \text{from the assumption } (\square) \\ &= \frac{k^2 + k + 2k + 2}{2} \\ &= \frac{(k+1)(k+2)}{2} \\ &= \frac{(k+1)((k+1)+1)}{2}. \end{aligned}$$

We're done! To recap a final time: we've shown that if someone tells us that $(*)$ is true for some $n = k$, then we can use this to establish that statement $(*)$ **must also** be true for $n = k+1$. This is the **fuse** we set up. Then all we needed to do to finish was set off the fuse with a **spark** by explicitly checking $(*)$ for $n = 1$.