Things to Think on Week 4

Definition. A *graph* is a collection of *vertices* some of which are connected by lines, called *edges*. A *simple graph* is one where any two vertices are connected by *at most* one edge and there are no loops. In this homework we will only deal with simple graphs. The *degree* of a vertex is the number of edges coming out of it.

1. A *complete graph* on n vertices, denoted by K_n, is the graph with n vertices where every pair of vertices is connected by a unique edge. For example, K_2 is a line segment, and K_3 is a triangle.

   (a) Draw pictures of K_4, K_5, and K_6. Come up with a guess for how many edges are in K_n (you should have a formula depending on n).

   (b) Prove your guess in part (a) in two different ways, and use your solution to prove that

   $$1 + 2 + 3 + \cdots + (n-1) = \frac{n(n-1)}{2}$$

2. A graph is called *connected* if any two vertices can be joined by a path (i.e. a sequence of edges where the endpoint of any edge is the starting point of the next.)

   (a) In the country of Seven there are 15 towns, each of which is connected to at least 7 others. Prove that one can always travel from any town to any other town by some sequence of roads.

   (b) Show, more generally, that any (simple) graph with n vertices, each of which has degree at least $(n-1)/2$, is connected. (Draw some examples!)

3. In a certain country, 100 roads lead out of each city, and it is possible to get from any city to any other. One road is closed for repairs. Is it still possible to get from any city to any other? If so, prove it; if not, construct a counterexample. (If you construct a counterexample that's not a simple graph, decide whether or not the claim holds for simple graphs. If not, produce another counterexample.)

4. Is it possible to draw 9 line segments in the plane so that each intersects exactly 3 others?