MATH CIRCLE HOMEWORK 7

1. In a simple version of "Deal or No Deal", there are five titanium cases numbered 1-5. Initially, each case contains a denomination of \$1 to \$5, respecting the number on the case.

(1) Howie comes in and moves the money around by switching the denominations between two cases at a time. For example, if he first switched the money in cases 1 and 2, and then switched the (new) money in cases 2 and 3, it would end up that Case 1 contains \$2, Case 2 contains \$3, and Case 3 contains \$1. Assuming the original initial configuration of the cases and money to start each of parts (a)-(c) below, discover the final configuration if he switches cases in the following order:
  (a) $1 \leftrightarrow 2,\ 1 \leftrightarrow 5,\ 3 \leftrightarrow 2,\ 4 \leftrightarrow 3,\ 5 \leftrightarrow 3,\ 1 \leftrightarrow 2,\ 3 \leftrightarrow 4$
  (b) $1 \leftrightarrow 2,\ 2 \leftrightarrow 3,\ 3 \leftrightarrow 4,\ 4 \leftrightarrow 5,\ 5 \leftrightarrow 1$
  (c) $4 \leftrightarrow 5,\ 5 \leftrightarrow 4,\ 1 \leftrightarrow 4,\ 4 \leftrightarrow 3,\ 3 \leftrightarrow 1,\ 1 \leftrightarrow 3,\ 3 \leftrightarrow 4,\ 4 \leftrightarrow 1$

(2) On the other hand, suppose now that Howie wants to end up with the money in the following configurations, only by successively switching the money in two cases at a time. In each part (a)-(c) below, determine if this is possible. If it is, give a way to do it. If not, explain why not.
  (a) Case 1: \$2 Case 2: \$3 Case 3: \$4 Case 4: \$5 Case 5: \$1
  (b) Case 1: \$3 Case 2: \$1 Case 3: \$2 Case 4: \$4 Case 5: \$4
  (c) Case 1: \$2 Case 2: \$1 Case 3: \$4 Case 4: \$5 Case 5: \$3

2. Describe the group of symmetries of a regular hexagon – we'll call it D_6.

(1) How many distinct symmetries are there total? Write down each symmetry in two-row notation which describes a permutation of the vertices, labelled 1-6. See the Permutations worksheet from class!

(2) What is the smallest number of symmetries needed to describe the whole group if we're allowed to build them up using the composition operation described in the Symmetry Groups worksheet from class?

3. Alice and Bob are playing a game. There is initially a stack of ten cards, numbered 1-10. Each player is randomly distributed five of the cards. Alice starts – they take turns playing a single card face up in a pile between them. Play continues back and forth until the sum of all the cards in the pile is divisible by 11. The last person to play a card is the winner.

Who has a winning strategy and what is it?