MATH CIRCLE HOMEWORK 5

1. Let p be a prime, and u be some integer not divisible by p. Show that if $ux \equiv uy \pmod{p}$ for some integers x and y, then $x \equiv y \pmod{p}$.

2. Let G be a planar graph with v vertices, e edges, and f faces. Recall that the unbounded outside of the graph is considered a face.

(a) Recall the definition of a **tree** (see the homework on `math.washington.edu/~mathcircle/circle/hw6.pdf` if you've forgotten). If G is a tree with v vertices, we proved in that homework assignment that G has $e = v - 1$ edges. Prove that in the case that G is a tree,

$$v - e + f = 2.$$

(b) Prove that the formula $v - e + f = 2$ holds even if G is not a tree. *Hint.* Try induction.

3. (From BAMO) In a plane, we are given line l, two points A and B neither of which lies on line l, and the reflection A_l of point A across line l. Using only a straightedge, construct the reflection B_l of point B across line l. Prove that your construction works. *Note.* "Using only a straightedge" means that you can perform only the following operations:

(a) Given two points, you can construct the line through them.
(b) Given two intersecting lines, you can construct their intersection point.
(c) You can select (mark) points in the plane that lie on or off objects already drawn in the plane. (The only facts you can use about these points are which lines they are on or not on).

4. Twenty points are placed around a circle. Two players take turns connecting two of the points with a line segment that does not cross any previously drawn line segments. The player who cannot do so loses. Who has the winning strategy, and what is it?