MATH CIRCLE HOMEWORK

1. What are the last 4 digits of 9999^{9999}?

2. Ina right triangle, the longest side is called the *hypotenuse* and the other two sides are called *legs*. Prove that if a right triangle has all integer side lengths and the hypotenuse has even length, then the length of both legs must also be even.

3. The divisibility rule for 3s and 9s involves looking at the sums of the digits. Discover and prove a similar divisibility rule for 11s.

4. I have encoded a secret message below, and it's your job to decrypt it. The encryption method I used is described as follows: to eahc letter of the alphabet we can associate an integer 0 through 25:

$$\mathbf{A} = 0 \quad \mathbf{B} = 1 \quad \mathbf{C} = 2 \quad \cdots \quad \mathbf{Z} = 25$$

I then piecked a key value $\alpha = 21$, and I changed each letter by multiplying its associated value by α, then reducing modulo 26.

For example, $\mathbf{Z} = 25$, and $\alpha \cdot 25 = 525 \equiv 5 \pmod{26}$. Since $\mathbf{E} = 5$, this means **Z** is changed to **E**.

(1) Here's the message to decode:

SAJR QMTQXG MO VANANAO!

(2) Re-encode the message you decoded above, this time using the value $\alpha = 2$. Do you foresee any problems if you were trying to now decode this message?
(3) For any ket α between 0 and 25 you'll be able to encode any message, but *decoding* may prove difficult for certain keys. Which values of α do you think will give you no trouble?
(4) (Challeng) Prove your guess from part (3).

5. (Challenge) The number 2 is drawn on a blackboard. two players take turns changing this number with the following rule: given a number x on the blackboard, a player may add any nonnegative number smaller than x to that number to obtain a new number which they then write on the blackboard in place of x. The player who reaches 1000 first wins. What is the winning strategy?