MATH CIRCLE WEEK 8

1. Prove using induction that for any positive integer n,

$$1 + 3 + 9 + 27 + \cdots + 3^n = \frac{1}{2}(3^{n+1} - 1).$$

2. Prove using induction that for any positive integer n,

$$1 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

3. Prove that for any integer $n \geq 3$, you can find n **distinct** positive integers $m_1, m_2, \ldots, m_n$ such that

$$\frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_n} = 1.$$

For example, if $n = 3$, we can choose $m_1 = 2$, $m_2 = 3$, and $m_3 = 6$:

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1.$$

4. While searching an abandoned pirate ship with a friend, you came across six pirate skeletons seated at a round table holding goblets in the air in a final toast. Each goblet had a gold piece labeled with the amount of money it was worth. You recognized the pirates from a storybook you had seen; their names, going clockwise around the table, were Captain Alice, First Mate Beetlejuice, Crazy Cynthia, Deafening Don, Evil Elizabeth, and Footstool Fred (who, even in the afterlife, was being used as a footstool by the captain).

A scroll on the table stated that you and your friend could take the gold, but you must do it in a specific way to avoid bursting into flames. You must each take turns, and follow these rules:

(1) If one player takes from Alice's or Don's goblet, then the other must take from a pirate sitting two places away on the next turn (so, for example, if you took from Alice's goblet, your friend must then take from either Cynthia's goblet or Elizabeth's goblet).

(2) If one player takes from Beetlejuice's or Elizabeth's goblet, the other must take from a pirate sitting either two or three spots away *clockwise* on the next turn.

(3) If one player takes from Cynthia's or Fred's goblet, then the other must take from a pirate sitting either two or three spaces away *counterclockwise* around the circle on the next turn.

In order to get the most money, do you go first or second? What is your strategy? (Hint: Draw a picture and this problem will become much more manageable!)