MATH CIRCLE WEEK 7

Some more definitions and facts about graphs.

Definition 1. An *Eulerian path* in a graph is a path that traverses every edge exactly once.

Definition 2. A *cycle* is a path that starts and ends at the same vertex. An *Eulerian cycle* is a cycle that traverses every edge exactly once.

Theorem 1. If a graph has an Eulerian path then it has no more than two vertices of odd degree.

We can interpret this as follows. Imagine the graph is drawn on a piece of paper. If the entire graph can be traversed without lifting the pencil from the paper while tracing each edge exactly once, then it has no more than two vertices of odd degree.

Theorem 2. If a graph has an Eulerian cycle then all its vertices have even degree.

Try to interpret this theorem in "plain English", similarly to the interpretation of Theorem 1 above.

1. Dylan just came home from Disneyland and told Chris that he saw an enchanted lake with three islands to each of which there led 1, 3 or 5 bridges, and that there was no way to get to the islands because no bridges connected the islands to the shore of the lake. Chris was sceptical about this. Do you think Dylan was wrong?

2. You have an incomplete set of dominos: all dominos with one or two empty squares are lost. Is it possible to arrange all dominos in a circle so that the adjacent squares have the same numbers on them? (A domino consists of two squares each one of which is marked by a number from 1 to 6. For each pair of numbers, including a pair of two repeating numbers, you have exactly one domino).

3. Suppose you are given a (connected) graph where each vertex has even degree.

(1) Show that you can put arrows on each edge such that, at each vertex, the number of incoming arrows is the same as the number of the outgoing arrows.
(2) Show further that it is possible to get from any vertex to any other vertex by following arrows *with the direction given.*

4. You have a piece of wire which is 120 in long.

(1) Is it possible to form a cube out of this wire with edges 10 in long? You are only allowed to bend the wire but not cut or break it.
(2) What is the smallest number of cuts one must make in the wire to form such a cube?