MATH CIRCLE HOMEWORK WEEK 7

This first page is full of some definitions.

A **graph** $G = (V, E)$ is a finite collection of vertices V and edges E. Each edge indicates two vertices which are connected.

Suppose u and v are vertices in a graph $G = (V, E)$. We say that there is a **path** from u to v if there is some way to start at u and end at v by following edges in G.

A graph is called **simple** if there is *at most* one edge between any two vertices, and there is no edge from a vertex to itself (no loops). All the graphs in this homework assignment are assumed to be simple graphs.

FIGURE 1. **Left:** An example of a graph which is not simple. We assume in this assignment that none of the graphs have loops like on the left vertex, or more than one edge between any two vertices. **Right:** An example of a simple graph.

The **degree** of a vertex in a simple graph is the number of edges connected to that vertex.

A graph is called **connected** if any two vertices are connected by a path. A graph is called a **tree** if it is connected, and moreover any two vertices are connected by *exactly* one path.

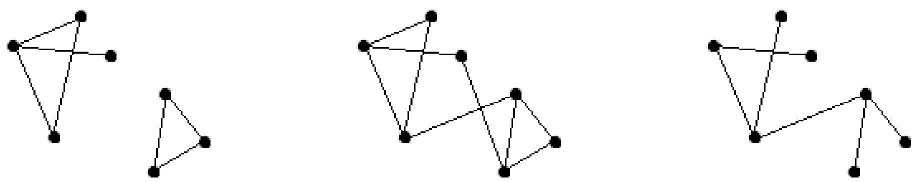

FIGURE 2. **Left:** An non-connected graph. **Middle:** A connected graph, but not a tree. **Right:** A tree.

Using the definitions on the first page, answer the following questions.

1. Prove that every (simple) graph has two vertices of the same degree.

2. A graph is called **complete** if, given any two vertices in the graph, there is an edge between them. If a complete graph has n vertices, how many edges does it have?

3. **(a)** Prove that in a connected graph with n vertices and exactly $n-1$ edges, there must be at least two vertices which have degree 1.

(b) Prove that a connected graph with n vertices and exactly $n-1$ edges is a tree.

4. Let G be a connected graph with n vertices. Show that the number of edges must be at least $n-1$, but no larger than $\frac{n(n-1)}{2}$.