MATH CIRCLE HOMEWORK WEEK 5

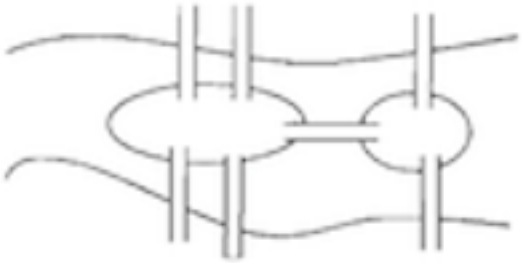
A note. Some of the problems for this week are a bit harder than usual (hopefully) because next week is Halloween and we won't be going over the homeworks until the week after, so you have a bit more time to think about these problems. Embrace the struggle- that is the fun part!

1. Consider a coordinate grid. We call the points with integer coordinates (x, y) a *lattice point*. Show that, given any collection of five lattice points, $\{A_1, A_2, A_3, A_4, A_5\}$, one of the line segments connecting one of the pairs, A_iA_j, of given points must have a midpoint that is another lattice point (not necessarily one of the five given).

2. Integers x and y are called *coprime* if they have no common divisors other than 1. Show that, given coprime integers x and y, we can always find integers a and b so that $ax - by = 1$.

3. Can a kingdom in which 3 roads lead out of each city have exactly 100 roads?

4. Below is a picture of a system of islands (including the two mainlands) and bridges. Is it possible to start at one island, cross every bridge exactly once, and end up at the island you started with? Is it possible to start at one island and cross every bridge exactly once without necessarily ending where you began?



5. Suppose we are given some number x (not necessarily an integer), and a positive integer N. Then show that we can always find two integers m and n so that $|nx - m| < \frac{1}{N}$.

6. Prove that the product of three consecutive numbers is always divisible by 6.

7. Find the smallest natural number n such that n! is divisible by 990.