HOMEWORK ON LOGIC AND PARITY

0. One day a prince came to the kingdom of Logos in search of a princess bride. There he met a king who, hoping to ensure that his daughter married the cleverest prince, arranged for a test. He led the prince to a hallway with 9 rooms, each with a sign on the door. He explained to the prince that exactly one of the rooms held the princess, and the other eight rooms were either empty or had a tiger inside them. The sign outside the room holding the princess told the truth, the sign outside rooms holding tigers told lies, and the signs outside the empty rooms could either be true or false. Here are the signs, numbered by their respective rooms:

(1) The princess is in an odd-numbered room.
(2) This room is empty.
(3) Sign 5 is right or sign 7 is wrong.
(4) Sign 1 is wrong.
(5) Either sign 2 or sign 4 is right.
(6) Sign 3 is wrong.
(7) The lady is not in room 1.
(8) This room contains a tiger and room 9 is empty.
(9) This room contains a tiger and sign 6 is wrong.

The prince studied the signs for a while and then said, "Well that's not fair, this is unsolvable! Give me a solid clue: tell me if room 8 is empty or not!"

So the king told him, and the prince was able to solve the puzzle.

Where is the princess?

1. The good people of Transylvania have enlisted your help in a very important trial. Two high-ranking officials were involved in a crime, but before the trial can take place, the court must figure out whether or not these two are humans or vampires and sane or insane. You see, in Transylvania, everyone is either a human or a vampire, and sane or insane.

The sane humans tell only the truth, while the insane humans tell only lies out of delusion. The sane vampires, on the other hand, tell lies out of malice, while the insane vampires tell only the truth (for they believe it to be a lie). These

officials are being very uncooperative. They have each agreed to answer two yes or no questions, and nothing more. Can you come up with two questions to ask that will allow you to pin down the species and mental health of both officials without a doubt?

2. The product of 22 integers is 1. Show that their sum cannot be zero.

3. A snail crawls through the plane at a constant velocity turning at right angles every 15 minutes. Show that the snail can only return to its starting position after a whole number of hours.

4. Nine numbers are placed in a circle: four 1's and five 0's. The following operation takes place: between each adjacent pair of numbers, one places a 0 if the numbers are different, and a 1 if they are the same, and then the old numbers are erased. Is it possible that some iteration of this operation results in all the remaining numbers being the same?

5. This puzzle also takes place in Transylvania. I will give you a dialogue between some logicians in Transylvania, but I will leave out some of what they said. It is your job to solve the puzzle with the missing information.

Three logicians were once discussing their separate trips to Transylvania.

"When I was there," said the first logician, "I met a Transylvanian named Igor. I asked him whether he was a sane human. Igor answered me [yes or no], but I couldn't tell from his answer what he was."

"That's a surprising coincidence," said the second logician, "I met that same Igor on *my* visit. I asked him whether he was a sane vampire and he answered me [yes or no], and I couldn't figure out what he was!"

"This is a double coincidence!" exclaimed the third logician. "I also met Igor and asked him whether he was an insane vampire and he answered me [yes or no], but I couldn't deduce what he was either."

Is Igor sane or insane? Is he human or vampire?

6*. [This problem is pretty tricky; try to finish all the other problems completely before thinking about this one.] A very strange boat crash lands onto a very strange cannibal island. The boat is strange because it has, within it, infinitely many pirates. To be precise, there is a pirate in the boat for each counting number $(1, 2, 3, \cdots)$. These pirates are captured by cannibals who have an unlimited supply of blue and red hats, and plan on lining up the pirates the next morning and placing, at random, a blue or red hat on each pirate. The pirates, having very good eyesight, will be able to see the sequence of colored hats stretching out to infinity in front of him, but will not be able to see his own hat or the hats of the (finitely many) pirates behind him. The cannibals will then start at the back of the line (it's very difficult to start at the front), and ask each pirate, in turn, what color hat they are wearing. Everyone who answers incorrectly will be eaten later that night, and everyone who answers correctly will be spared. Oh... and something

rather unfortunate is true. All of these pirates are deaf, and cannot hear the other answers. Nor can they communicate in any way during that morning. The pirates have all night to make a plan; what should they do in order to guarantee that all but at most finitely many of them survive? (These pirates are very gifted, and can do and memorize infinitely many things before the night is through).