UW Math Circle

Week 13

1 A peculiar game

Problem 1. Take 6 beans and arrange them into some number of piles (whatever you like). Now, do the following step over and over again: Make a new pile by taking 1 bean from each existing pile. After doing this step many times, what happens? (This game is called *Bulgarian solitaire*.)

Problem 2. What are all of the ways that you can divide 6 beans into piles? (A pile consists of 1 or more beans. Try to do this in a systematic way so that you're sure that you didn't miss any possibilities.)

Problem 3. Draw a diagram to prove that when starting with 6 beans, you always end up with the result you found in Problem 1, no matter how you split up the beans to start with.

Problem 4. (Bonus) Can you find a way to compute the number of ways to split n objects into piles efficiently? ("Efficiently" means without needing to list them all, but it still might take a few steps. Hint: first think about the number of ways to split n objects into exactly k piles.)

Stop here. Request the next page from your instructor when your group is done.

2 Different amounts, different results

Problem 5. Let's investigate how Bulgarian solitaire changes depending on the number of starting beans. Like before, feel free to start by splitting the beans into piles however you like. Work with your group to fill in the following table. (Consider having each group member try different rows.)

# of beans	What happens?
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

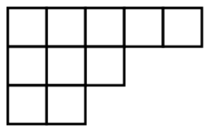
Problem 6. Guess a pattern to describe what will happen as you keep increasing the number of beans. Using the pattern you guessed, what will happen for 20 beans? 21 beans?

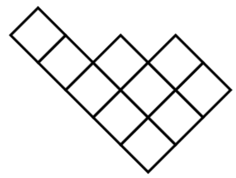
Stop here. Request the next page from your instructor when your group is done.

3 The old tool called Young diagrams

Why does your pattern work? Let's investigate this now.

In 1900, Alfred Young decided to study the different ways you can split n objects into piles using what we now call a Young diagram. They look like the picture on the left, which shows splitting 10 into piles (rows) of 5, 3, and 2. Since we don't care about the order of the piles, we always draw the bigger piles on top. For our purposes today, it will be useful to rotate Young diagrams to look like the picture on the right.





Problem 7. Try playing Bulgarian solitaire with rotated Young diagrams. Can you describe what each step means visually? After playing for a long time, can you generally describe what the ending configurations look like?

Problem 8. A *triangular number* is a number that can be expressed as a sum $1+2+3+\ldots+k$. For example, the first three triangular numbers are 1, 3 $(1+2)$, and 6 $(1+2+3)$.

Let n be the number of beans you start with, and let k be the smallest number such that $n \leq 1+2+3+\ldots+k$. Using your observations from the previous problem, explain why:

1. If n is triangular number, Bulgarian solitaire ends in a fixed configuration with piles of size $1, 2, 3, \ldots, k$.
2. If n is not a triangular number, Bulgarian solitaire ends in a cycle of length at most k.

Stop here. Request the next page from your instructor when your group is done.

4 Bonus problems

Problem 9. For what amounts of starting beans is it possible to end up in at least two different cycles, depending on the starting configuration of piles? What is the smallest number of starting beans for which there are 2 distinct ending cycles? 3 distinct ending cycles?

Problem 10. Some configurations of piles can be reached from performing a step of Bulgarian solitaire on another configuration, but some configurations cannot be reached - they can only be starting configurations. How many configurations cannot be reached from other configurations?

Problem 11. Since we are playing Bulgarian solitaire, in this map of Europe, find Bulgaria. If you have extra time, try to name as many other countries in the map as you can.