UW Math Circle

Week 12

1. What time of day will it be 100 hours from now? What day of the week will it be 100 days from now? If you rotate this paper by 900 degrees, what will it look like?

2. For each of the below equations, try to find integers b and r that make the equation true. Make the integer r as small as possible.

   (a) $100 = b \cdot 7 + r$.

   (b) $100 = b \cdot 24 + r$.

   (c) $900 = b \cdot 360 + r$.

3. For each of the below equations, try to find integers b and r that make the equation true. Make the integer r as small as possible.

   (a) $2^2 = b \cdot 4 + r$

   (b) $3^2 = b \cdot 4 + r$

   (c) $4^2 = b \cdot 4 + r$

   (d) $5^2 = b \cdot 4 + r$

4. For each of the below equalities, can you find integers n and b that makes the equality true? If not, argue that it is impossible.

   (a) $n^2 = b \cdot 4 + 0$

   (b) $n^2 = b \cdot 4 + 1$

   (c) $n^2 = b \cdot 4 + 2$

   (d) $n^2 = b \cdot 4 + 3$

5. Consider the number $1 \cdot 2 \cdot 3 \cdot 4 = 24$. How many times does 24 contain the prime factor 2? What about 3?

6. Now consider the number $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720$. How many times does 720 contain the prime factor 2? What about 3? What about 5?

7. Consider $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10$, which we call "10 factorial" and write as "10!". For each of the prime factors of 10! count how many times 10! contains that prime factor.

8. Try to find a general formula for the above problem. That is, if n is a positive integer and p is a prime number, how many times does $n!$ contain the prime factor p?

9. You want to send a letter, but all your have are 4 and 5 cent stamps. Can you use these stamps to put the right amount of postage on your letter if the postage is 10 cents? 11 cents? 12 cents? What is the largest amount of postage that you cannot make with 4 and 5 cent stamps?

10. Now you have 3 and 5 cent stamps. What is the largest amount about postage you cannot make? What about with 7 and 11 cent stamps?

11. Formulate a conjecture for the maximum amount of postage you can't make with a and b cent stamps, where a and b are positive and their greatest common divisor is 1. Can you prove your conjecture is correct?

12. For each of the equalities below, try to find integers a and b to make the equality true.

   (a) $13 = a^2 + b^2$.

   (b) $25 = a^2 + b^2$.

   (c) $74 = a^2 + b^2$

   (d) $80 = a^2 + b^2$.

13. If there are integers a and b so that $n = a^2 + b^2$ we say n is *the sum of two squares.* Are any of the numbers 3, 7, 11, 15, 19, 23 the sum of two squares? Why or why not? Is 2023 the sum of two squares?

14. If you are not limited to only two squares, can you write any number as the sum of squares? (For example $31 = 2^2 + 3^2 + 3^2 + 3^2$ uses 4 squares). What if you are limited to 100 squares? What if you are limited to 10? How low can you make this limit.

15. Try to identify which numbers are the sum of two squares.