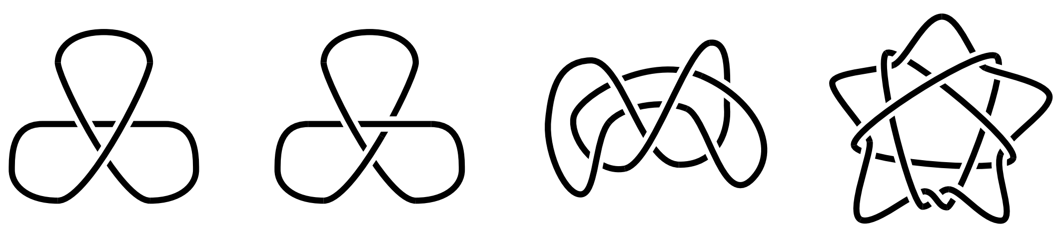
UW Math Circle

Week 8

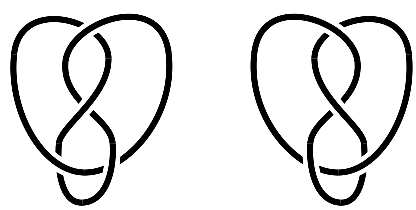
Today, we will learn about knot theory. Before getting started, get a piece of string from an instructor.

In math, a *knot* is what you get by taking one piece of string, possibly twisting it in some complicated way, and then tying the ends together to form a loop. (Warning: this is different than how we usually use the word “knot” in English – your shoelaces are not knots because there are dangling ends that don’t form loops!)

1. Tie the ends of your piece of a string to make a circle. Without untying the ends, can you make it look like the below pictures? Pay attention to how the strands go over/under each other.



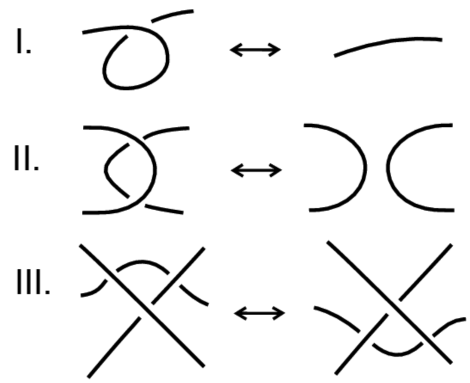
2. We say that two drawings of knots are *equivalent* if you can move the strands around to get from one to the other without untying the loops. Show that the following two knots are equivalent.



3. Find an example of a knot which is *not* equivalent to its mirror image.

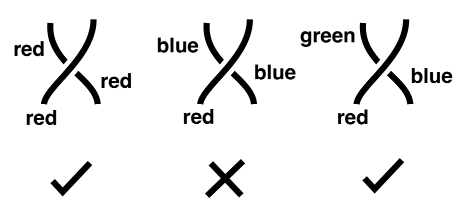
To show that two knots are equivalent, just show the sequence of moves. But you might have noticed it’s difficult to say exactly why two knots are different. How do you tell the difference between really impossible, or just not trying hard enough? Let’s investigate one way to do this.

4. In 1927, Kurt Reidemeister realized that two *drawings* of a knot represent equivalent knots if and only if you can transform one to the other using the following 3 kinds of moves:



Go back to problems 1 and 2 and explain them using Reidemeister moves. Draw the intermediate pictures.

5. Pick three colors. Say that a drawing of a knot is *tricolorable* if you can assign a color to every unbroken segment in the drawing such that all three colors are used, and at every intersection, either all segments have the same color, or all three colors appear. (The printer is black and white, but you get the idea.)



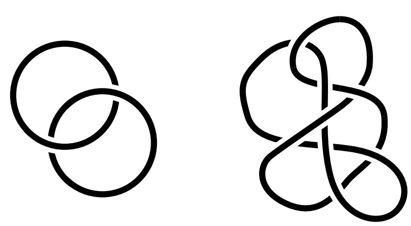
Which knots from question 1 are tricolorable? Can you guess how you can test whether two knots are equivalent using tricolorability?

6. Show that if a knot is tricolorable, and then you apply any of the Reidemeister moves, then the resulting knot is also tricolorable. (When a property doesn't change after applying Reidemeister moves, we call it *invariant*.) Use this to explain your guess from the previous problem.

7. Explain why tricolorability is not enough to distinguish the two knots you came up with problem 3. (So tricolorability works to distinguish some knots, but not all! Mathematicians have invented more fancy tools for this with fancy names like *Kaufmann bracket* or *Jones polynomial* that can distinguish more kinds of knots.)

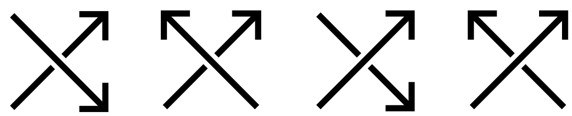
A *link* is a collection of knots which may be entangled together.

8. Make the link on the left with some string, and then show that it is equivalent to the link on the right. (Feel free to explain with either Reidemeister moves or physical moves.)



9. Make a link using two circles that is not equivalent to the link from problem 8.

10. Again, it can be a little tricky to explain why two links are not equivalent, so let's use an invariant. First, give every knot in the link a direction. To find the *linking number* of an intersection, find the top strand and point along its direction. Turn your hand counterclockwise until your finger is over the bottom strand. If you are pointing along the direction assigned to the bottom strand, the linking number is 1. If you are pointing in the opposite direction assigned to the bottom strand, the linking number is -1. What is the linking number for the following four intersections?



The linking number of a whole link is the sum of linking numbers for every intersection between two different knots. Compute the linking numbers for the three drawings in problem 8 and 9.

11. Show that the linking number is invariant under Reidemeister moves, so it can be used to prove that two links are different.

12. Come up with an example of two links that are different, but have the same linking number. (So again, it only works to distinguish some links, but not all.)