Week 9

Question 1. Count some things!

(a) How many ways can you choose a subset of the numbers $1, 2, \ldots, n$ that does not contain a pair of consecutive numbers? (For example: $1, 3, 5, 6, 9$ does not qualify because it contains 5 and 6.)

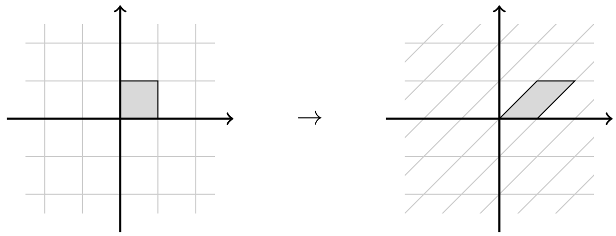
(b) How many binary numbers are there between 0 and $111 \cdots 1$ (with n ones) that do not contain two 1 digits in a row?

(c) How many ways can you make a list of 1's and 2's that adds up to n? (The lists $1, 2$ and $2, 1$ count as different lists.)

(d) How many ways can you make a list of odd numbers that adds up to n? (Count $1, 3$ and $3, 1$ as different lists.)

(e) How many ways can you make a list of numbers that are larger than 1 and add up to n? (Again, $2, 3$ and $3, 2$ count as different lists.)

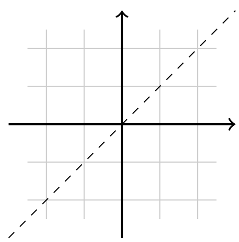
Question 2. This is a "shear transformation" of the 2D plane:



Each point at height y is moved y units to the right.
What happens to the following coordinates if we do a shear transformation to them?

(a) $(1,1)$
(b) $(-1,2)$
(c) $(3,0)$
(d) $(2,-1)$
(e) $(7,12)$
(f) (x,y)

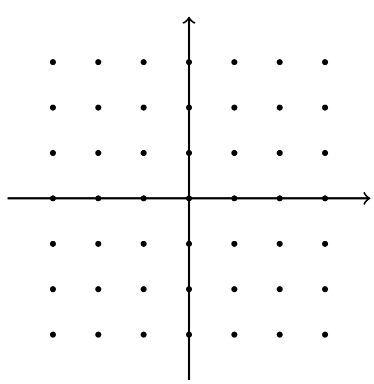
Question 3. Suppose we take the 2D plane and reflect it along this line:



Where do these points end up?

(a) $(2,1)$
(b) $(0,2)$
(c) $(1,-2)$
(d) $(4,-1)$
(e) $(7,12)$
(f) (x,y)

Question 4. On this picture, draw an arrow from each point to where it ends up under:
(a) the shear transformation,
(b) the reflection.



Question 5. Now, what if we do the shear transformation and then also the reflection!

Where do these points end up after doing both transformations (shear first, then reflection)?

(a) $(0,0)$

(b) $(2,0)$

(c) $(0,2)$

(d) $(3,1)$

(e) $(6,2)$

(f) (x,y)

Question 6. What about these points?

(a) $(1,1)$

(b) $(1,2)$

(c) $(2,3)$

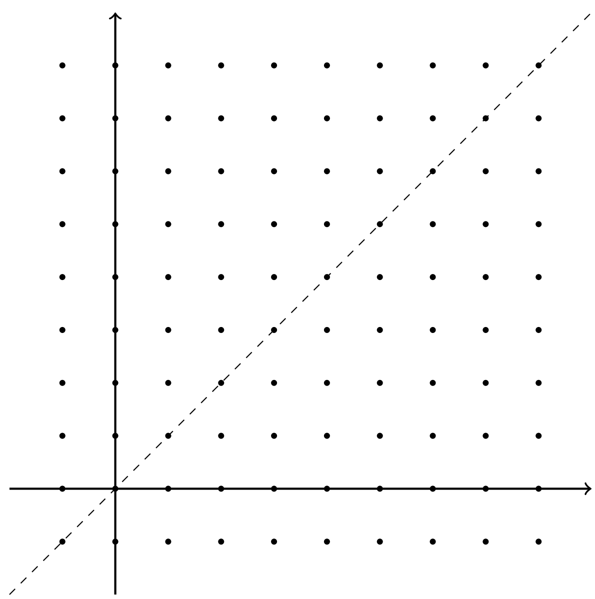
(d) $(3,5)$

(e) $(5,8)$

(f) $(8,13)$

(Do you recognise these numbers. . . ?)

Question 7. Like on the previous page, draw an arrow from each point to where it ends up after doing both transformations:

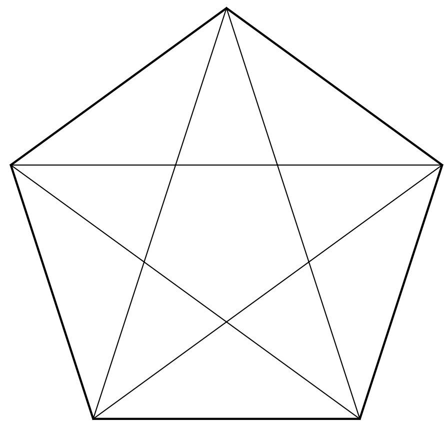


Question 8. Pick a point, follow its arrow to a new point, then follow the arrow at that point, and so on. What's the general long-term behaviour if you keep following arrows like this? Do different starting points do different things?

Question 9. Can you find a point (x,y) that ends up at a scalar multiple of itself (that is, it ends up at (cx,cy) for some number c)? These points might not have integer coordinates.

Draw these points on the graph above.

Question 10. Consider this picture:



Suppose the outer pentagon has sides of length 1.

What are the lengths of:

(a) The diagonals of the outer pentagon?

(b) The sides of the inner, upside-down pentagon?

(c) The edges of the “star” shape?

(Hint: Two triangles are called “similar” if one is a scaled-up copy of the other. Equivalently, the angles of one triangle are equal to the angles of the other. If two triangles are similar, the ratios between their edge lengths are equal.)

Question 11. Let F_n mean "the nth Fibonacci number" (so $F_0 = 0,\ F_1 = 1,\ F_2 = 1,\ F_3 = 2$ and so on). Explain the following Fibonacci identities:

(a) $F_0 + F_1 + \cdots + F_n = F_{n+2} - 1$

(b) $F_0 + F_2 + F_4 + \cdots + F_{2n} = ???$

(c) $F_1 + F_3 + F_5 + \cdots + F_{2n-1} = ???$

(d) $F_n^2 + F_{n+1}^2 = ???$

(e) $F_0^2 + F_1^2 + \cdots + F_n^2 = F_n F_{n+1}$

(f) $3F_n = F_{n+2} + F_{n-2}$

(g) $\binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots = F_n$

(h) $F_{2n-1} = \binom{n}{1} F_0 + \binom{n}{2} F_1 + \cdots + \binom{n}{n} F_{n-1}$.